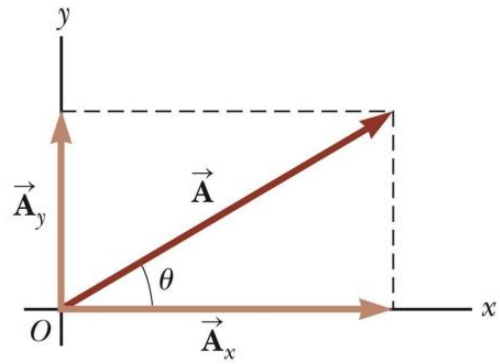
2- Vectors



Components of a Vector, Introduction A component is a projection of a vector along an axis.

- ❖ Any vector can be completely described by its components.

It is useful to use rectangular components.

- ❖ These are the projections of the vector along the x- and y-axes.

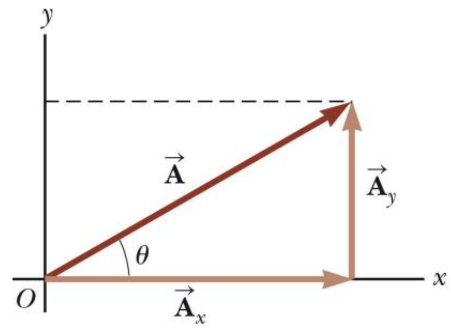
Vector Component Terminology

$\vec{A}_x$ and $\vec{A}_y$ are the component vectors of $\vec{A}$.

- ❖ They are vectors and follow all the rules for vectors.

A_x and A_y are scalars, and will be referred to as the components of A.

Components of a Vector



Assume you are given a vector $\vec{A}$. It can be expressed in terms of two other vectors $\vec{A}_x$ and $\vec{A}_y$

$\vec{A} = \vec{A}_x + \vec{A}_y$

These three vectors form a right triangle.

The x-component of a vector is the projection along the x-axis.

$A_x = A \cos\theta$

The y-component of a vector is the projection along the y-axis.

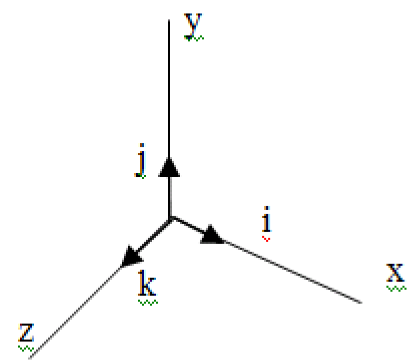
$A_y = A \sin\theta$

This assumes the angle θ is measured with respect to the x-axis.

The components are the legs of the right triangle whose hypotenuse is the length of A

$A = \sqrt{Ax^2 + Ay^2}$ and $\theta = \tan^{-1} \frac{Ay}{Ax}$

2-1 *Vectors collection*



$\vec{A} = \vec{A}_x\, i + \vec{A}_y\, j + \vec{A}_z\, k$

$\vec{B} = \vec{B}_x\, i + \vec{B}_y\, j + \vec{B}_z\, k$

$\vec{A} + \vec{B} = (\vec{A}_x + \vec{B}_x)\, i + (\vec{A}_y + \vec{B}_y)\, j + (\vec{A}_z + \vec{B}_z)\, k$

2-2 *Subtract vectors*:

$\vec{A} - \vec{B} = (\vec{A}_x - \vec{B}_x)\, i + (\vec{A}_y - \vec{B}_y)\, j + (\vec{A}_z - \vec{B}_z)\, k$

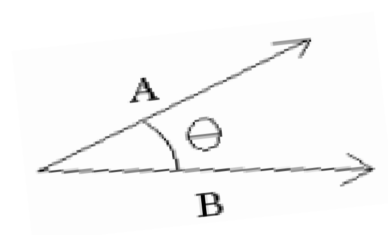
To find the numerical value of the vector A can be:

$|\vec{A}| = \sqrt{Ax^2 + Ay^2 + Az^2}$

And

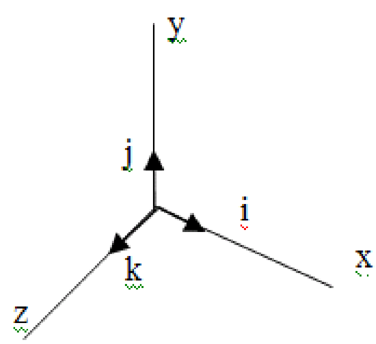
Commutative Law for Addition: A + B = B+A

2-1 Scalar Product or dot (•) product



The scalar product of two vectors, A and B denoted by A•B, is defined as the product of the magnitudes of the vectors times the cosine of the angle between them, as illustrated in Figure.

$\vec{A} \cdot \vec{B} = \vec{A}_x \vec{B}_x + \vec{A}_y \vec{B}_y + \vec{A}_z \vec{B}_z$



i = (1,0,0)	i.i =1	i•j = 0
j = (0,1,0)	j.j =1	j•k= 0
k = (0,0,1)	k•k=1	k•i= 0

$\vec{A} \cdot \vec{B} = AB\cos\theta$

$\cos\theta = \frac{|A.B|}{|A|\,|B|}$

***Note that** the result of a dot product is a scalar, not a vector.*

The rules for scalar products are given in the following list:

1- $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$ -------(Prof that)

$$\vec{A} \bullet \vec{B} = \vec{A}_x \vec{B}_x + \vec{A}_y \vec{B}_y + \vec{A}_z \vec{B}_z$$
$$= \vec{B}_x \vec{A}_x + \vec{B}_y \vec{A}_y + \vec{B}_z \vec{A}_z$$
$$\therefore \vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

2- $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

3- $m (\vec{A} \bullet \vec{B}) = (m\vec{A}) \bullet \vec{B}$
$$= \vec{A} \bullet (m\vec{B}) = (\vec{A} \bullet \vec{B})\, m$$

<u>prove the cosine law</u>:

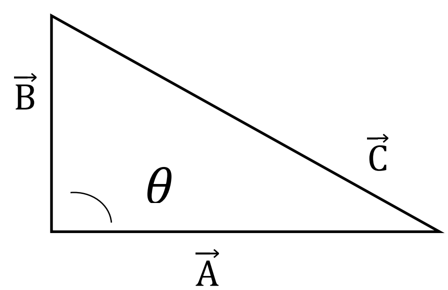
$\vec{C} = \vec{A} + \vec{B}$

$\vec{C} \bullet \vec{C} = (\vec{A} + \vec{B}) \bullet (\vec{A} + \vec{B})$

$C^2 = A \bullet A + A \bullet B + B \bullet A + B \bullet B$

$C^2 = A^2 + B^2 + 2A \bullet B$

$C^2 = A^2 + B^2 + 2AB \cos\theta$:



Example2

Let us do an example. Consider two vectors $\vec{A} = 2i + 2j$ and $\vec{B} = 6i - 3j$
Now what is the angle between these two vectors?

Solution: From the definition of scalar products, $\vec{A} \bullet \vec{B} = AB\cos\theta$ we have

$\vec{A} \bullet \vec{B} = |A|\,|B| \cos\theta$

$$\therefore \cos\theta = \frac{|A.B|}{|A|\,|B|}$$

$\vec{A} \bullet \vec{B} = 2i + 2j.\ 6i - 3j = 6$

$|A| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2.83$

$|B| = \sqrt{6^2 + (-3)^2} = \sqrt{45} = 6.71$

$$\therefore \cos\theta = \frac{6}{2.83x6.71} = 0.316 \qquad \therefore \theta = \cos^{-1} 0.316 = 71^{\circ}.36'$$

Question 1: Prove that the two vectors are perpendicular if: A•B = 0?

Solutions:

$\vec{A} \bullet \vec{B} = |A|\,|B| \cos\theta$

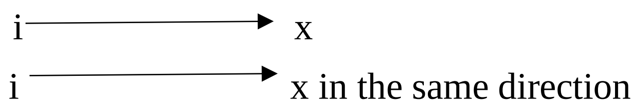
$\therefore \cos\theta = \frac{|A.B|}{|A|\,|B|} = \frac{0}{|A|\,|B|} = 0$

$\therefore \theta = \cos^{-1}(0) = 90°$

$\therefore A \perp B$

Question 2: Prove that: **i.i** = **j.j** = **k.k** = 1

Solution: i.i = ii cos θ

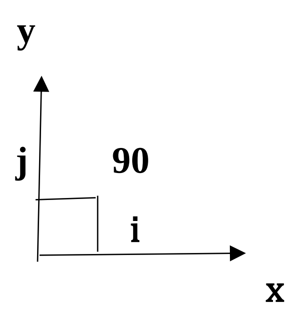


That is, the angle between them is = 0

∵ **i•i** = (1,0,0) (1,0,0) cos θ

i•i = (1,0,0) (1,0,0) •1 =1 so on

Question 3: Prove that: **i•j = j.k = k.j = 0**



Solution

i•j = i j cos θ

= (1,0,0) (0,1,0) cos 90

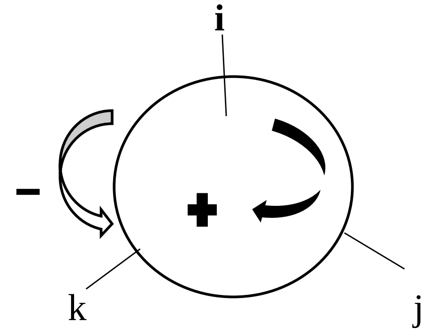
∵ As the angle between them = 90°

Cos 90 = 0

∴ i•j = 0

2-2 Vector Product or ***<u>Cross (x)</u>*** Product

$\vec{A} \times \vec{B} = AB \sin \boldsymbol{\theta} \Longrightarrow \sin \boldsymbol{\theta} = \frac{|A \times B|}{|A|\,|B|}$



i x i = 0	i x j = k	i x k = -j
j x j = 0	j x k = I	k x j = -i
k x k= 0	k x i = j	j x i = - k

$\vec{A} \times \vec{B} = - \vec{B} \times \vec{A}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ Ax & A_y & A_z \\ Bx & By & Bz \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\, i + (A_z B_x - A_x B_z)\, j + (A_x B_y - A_y B_x)$$

The vector product or the cross product multiplies two vectors in such a way that the resultant is a new vector.

If $\vec{A} = \vec{A}_x\, i + \vec{A}_y\, j + \vec{A}_z\, k$ and $\vec{B} = \vec{B}_x\, i + \vec{B}_y\, j + \vec{B}_z\, k$ then let $\vec{C} = \vec{C}_x\, i + \vec{C}_y\, j + \vec{C}_z\, k$ be the result of this multiplication.

Let A=i+2j+3k; B=2i-3j+k. Then the cross product of A and B is

$\vec{C} = \vec{A} \times \vec{B} = |A|\,|B| \sin \theta$

$$\vec{C} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -3 & 1 \end{vmatrix} = (2*1-3*(-3))\, i + (3*2-1*1)\, j + (-3*1- 2*2)\, k$$

$\vec{C} = 11i + 5j - 7k$

Question4: If you know that A = 2i +2j – k and B = i – j +2k,

Find: 1) A x B , 2) A **.** B 3) and the angle between them?

Solutions:

1) A **.** B = 2- 2 -2 = -2

2)A x B = $\begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$ = (4-1) i + (-1-4) j + (-2-2) k

A x B = 3i -5j – 4k

3) A= $\sqrt{2^2 + 2^2 + (-1)^2}$ = $\sqrt{4 + 4 + 1}$= $\sqrt{9}$ = 3

B =$\sqrt{1^2 + 1^2 + 2^2}$ = $\sqrt{1 + 1 + 4}$ = $\sqrt{6}$

Cos θ = $\frac{|A.B|}{|A|\,|B|} = \frac{-2}{3\sqrt{6}}$ ∴ θ = $\cos^{-1} \frac{-2}{3\sqrt{6}}$

Homework

Q1) Prove that i x j = k , j x k = i and k x i = j ?

Q2) Find the unit vector perpendicular to the plane between the two vectors

$\vec{A}$ = 2i + 2j – k and $\vec{B}$ = i - j +2k

Q3) Find the angle between the two vectors A = 2i + 2j -k and B = 6i -3j +2k?