Applications of Group Theory

References:

- Introduction to Modern Abstract Algebra, by David M. Burton.
- Groups and Numbers, by R. M. Luther.
- A First Course in Abstract Algebra, by J. B. Fraleigh.
- Group Theory, by M. Suzuki.
- Abstract Algebra Theory and Applications, by Thomas W. Judson.
- Abstract Algebra, by I. N. Herstein.
- Basic Abstract Algebra, by P. B. Bhattacharya, S. K. Jain and S. R. Nagpaul.

1. The Jordan-Holder Theorem and Related Concepts.

Definition(1-1):

By a *chain* for a group $(G,*)$ is meant any finite sequence of subsets of

$G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$ descending from G to $\{e\}$ with the property that all the pairs $(H_i,*)$ are subgroups of $(G,*)$.

Remark(1-2):

The integer n is called the length of the chain. When $n = 1$, then the chain in definition (1-1) will called the trivial.

Example(1-3):

Find all chains in a group $(Z_4, +_4)$.

Solution: The subgroups of a group $(Z_4, +_4)$ are :

- $H_1 = (Z_4, +_4)$
- $H_2 = (\{0\}, +_4)$
- $H_3 = (\langle 2 \rangle, +_4) = (\{0,2\}, +_4)$

The chains of a group $(\mathrm{Z}_4, +_4)$ are

$$\mathrm{Z}_4 \supset \{0\} \text{ is a chain of length one}$$

$$\mathrm{Z}_4 \supset \langle 2 \rangle \supset \{0\} \text{ is a chain of length two.}$$

Example(1-4):

In the group $(\mathrm{Z}_{12}, +_{12})$ of integers modulo 12, the following chains are normal chains:

$$\mathrm{Z}_{12} \supset \langle 6 \rangle \supset \{0\},$$

$$\mathrm{Z}_{12} \supset \langle 2 \rangle \supset \langle 4 \rangle \supset \{0\},$$

$$\mathrm{Z}_{12} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \{0\},$$

$$\mathrm{Z}_{12} \supset \langle 2 \rangle \supset \langle 6 \rangle \supset \{0\}.$$

All subgroups are normal, since $(\mathrm{Z}_{12}, +_{12})$ is a commutative group.

Definition(1-5): (***Normal Chain***)

If $(H_i,*)$ is a normal subgroup of a group $(G,*)$ for all $i = 1, \dots, n$, then the chain $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$ is called a *normal chain*.