Corollary(1-20):

The group $({}^{G}/_{H}, \otimes)$ is a simple, if $\left|{}^{G}/_{H}\right|$ is a prime number.

Examples(1-21);

1. Show that $(\langle 2\rangle, +_{12})$ is a maximal normal subgroup of $(\mathrm{Z}_{12}, +_{12})$.
2. Show that $(\langle 3\rangle, +_{15})$ is a maximal normal subgroup of $(\mathrm{Z}_{15}, +_{15})$. (**Homework**)

Solution(1): $(\langle 2\rangle, +_{12}) = (\{0,2,4,6,8,10\}, +_{12})$

$\left|{}^{G}/_{H}\right| = \frac{|G|}{|H|} = \frac{|\mathrm{Z}_{12}|}{|\langle 2\rangle|} = \frac{12}{6} = 2$ is a prime $\Rightarrow \frac{\mathrm{Z}_{12}}{\langle 2\rangle}$ is a simple (by Corollary (1-20)). From Theorem (1-19), we get that $\langle 2\rangle$ is a maximal normal subgroup of Z_{12}.

Corollary(1-22):

A normal chain $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$ is a composition of a group $(G,*)$, if $({}^{H_i}/_{H_{i-1}}, \otimes)$ is a simple group for all $i = 1, \dots, n$.

Example(1-23);

Show that $Z_{60} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \langle 12 \rangle \supset \{0\}$ is a composition chain of a group $(Z_{60}, +_{60})$.

Solution: $\frac{|Z_{60}|}{|\langle 3 \rangle|} = \frac{60}{20} = 3$ is a prime $\Rightarrow \frac{Z_{60}}{\langle 3 \rangle}$ is a simple.

So, we get that $\langle 3 \rangle$ is a maximal normal subgroup of Z_{60}.

$\frac{|\langle 3 \rangle|}{|\langle 6 \rangle|} = \frac{20}{10} = 2$ is a prime $\Rightarrow \frac{\langle 3 \rangle}{\langle 6 \rangle}$ is a simple.

So, we get that $\langle 6 \rangle$ is a maximal normal subgroup of $\langle 3 \rangle$.

$\frac{|\langle 6 \rangle|}{|\langle 12 \rangle|} = \frac{10}{5} = 2$ is a prime $\Rightarrow \frac{\langle 6 \rangle}{\langle 12 \rangle}$ is a simple.

So, we get that $\langle 12 \rangle$ is a maximal normal subgroup of $\langle 6 \rangle$.

$\frac{|\langle 12 \rangle|}{|\{0\}|} = \frac{5}{1} = 5$ is a prime $\Rightarrow \frac{\langle 12 \rangle}{\{0\}}$ is a simple.

So, we get that $\{0\}$ is a maximal normal subgroup of $\langle 12 \rangle$.

By corollaries (1-19) and (1-21), we have that $Z_{60} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \langle 12 \rangle \supset \{0\}$ is a composition chain of a group $(Z_{60}, +_{60})$.

Theorem(1-24):

Every finite group $(G,*)$ with more than one element has a composition chain.

Theorem(1-25): (**Jordan-Holder**)

In a finite group $(G,*)$ with more than one element, any two composition chains are equivalent.

Example(1-26):

In a group $(Z_{60}, +_{60})$, show that the two chains

$$Z_{60} \supset \langle 3 \rangle \supset \langle 6 \rangle \supset \langle 12 \rangle \supset \{0\}$$

$$Z_{60} \supset \langle 2 \rangle \supset \langle 6 \rangle \supset \langle 30 \rangle \supset \{0\},$$

are compositions and equivalent.

Solution:

$(^{Z_{60}}/_{\langle 3 \rangle}, \otimes) \cong (^{\langle 2 \rangle}/_{\langle 6 \rangle}, \otimes)$, since $\left|^{Z_{60}}/_{\langle 3 \rangle}\right| = \frac{60}{20} = 3 = \left|^{\langle 2 \rangle}/_{\langle 6 \rangle}\right| = \frac{30}{10}$,

$(^{\langle 3 \rangle}/_{\langle 6 \rangle}, \otimes) \cong (^{Z_{60}}/_{\langle 2 \rangle}, \otimes)$, since $\left|^{\langle 3 \rangle}/_{\langle 6 \rangle}\right| = \frac{20}{10} = 2 = \left|^{Z_{60}}/_{\langle 2 \rangle}\right| = \frac{60}{30}$,

$(\langle 6\rangle/\langle 12\rangle, \otimes) \cong (\langle 30\rangle/\{0\}, \otimes)$, since $\left|\langle 6\rangle/\langle 12\rangle\right| = \frac{10}{5} = 2 = \left|\langle 30\rangle/\{0\}\right| = \frac{2}{1}$,

$(\langle 12\rangle/\{0\}, \otimes) \cong (\langle 6\rangle/\langle 30\rangle, \otimes)$, since $\left|\langle 12\rangle/\{0\}\right| = \frac{5}{1} = 5 = \left|\langle 6\rangle/\langle 30\rangle\right| = \frac{10}{2}$.

Therefore, by Jordan-Holder theorem the two chains

$$Z_{60} \supset \langle 3\rangle \supset \langle 6\rangle \supset \langle 12\rangle \supset \{0\}$$

$$Z_{60} \supset \langle 2\rangle \supset \langle 6\rangle \supset \langle 30\rangle \supset \{0\},$$

are compositions and equivalent.