**Theorem(4-5):**

Every subgroup of a solvable group is a solvable.

**Proof:** let $(H,*)$ be a subgroup of $(G,*)$ and $(G,*)$ is a solvable group.

To prove $(H,*)$ is a solvable.

Since $G$ is a solvable $\Longrightarrow$

there is a finite collection of subgroups of $(G,*)$, $G_0, G_1, \dots, G_n$ such that

1. $G = G_0 \supset G_1 \supset \cdots \supset G_{n-1} \supset G_n = \{e\}$,
2. $G_{i+1} \Delta G_i \quad \forall i = 0, \dots, n-1$,
3. ${}^{G_i}/_{G_{i+1}}$ is a commutative group $\forall i = 0, \dots, n-1$.

Let $H_i = H \cap G_i, \quad i = 0, \dots, n$

$H_0 = H \cap G_0, H_1 = H \cap G_1, \dots, H_n = H \cap G_n = \{e\}$

Each $H_i$ is a subgroup of $(G,*)$.

1. $G = H_0 \supset H_1 \supset \cdots \supset H_{n-1} \supset H_n = \{e\}$ is hold
2. $H_{i+1} \Delta H_i \quad \forall i = 0, \dots, n-1, \qquad H_i = H \cap G_i, \; H_{i+1} = H \cap G_{i+1}$, since $G_{i+1} \Delta G_i \Longrightarrow H_{i+1} \Delta H_i$

3. To prove ${}^{H_i}/_{H_{i+1}}$ is a commutative group $\forall i = 0, \dots, n-1$.

Let $f_i: H_i \longrightarrow {}^{G_i}/_{G_{i+1}}, i = 0, \dots, n-1$ such that $f_i(x) = x * G_{i+1} \forall x \in H_i \subseteq G_i$.

To prove $f_i$ is a homomorphism,

$f_i(x * y) = f_i(x) \otimes f_i(y)$ ?

$f_i(x * y) = x * y * G_{i+1} = (x * G_{i+1}) \otimes (y * G_{i+1}) = f_i(x) \otimes f_i(y)$

So, $f_i$ is a homomorphism

$f_i$ is onto ?

$R_{f_i} = \{f_i(x) : x \in H_i\} = \{x * G_{i+1} : x \in H_i\} = f_i(H_i)$

$\neq {}^{G_i}/_{G_{i+1}}$

$f_i(H_i) \subseteq {}^{G_i}/_{G_{i+1}} \Longrightarrow f_i$ is not onto

${}^{H_i}/_{\ker f_i} \cong f_i(H_i)$ ( by theorem of homomorphism)

$$\ker f_i = \{x \in H_i : f_i(x) = e'\} = \{x \in H_i : x * G_{i+1} = G_{i+1}\}$$
$$= \{x \in H_i : x \in G_{i+1}\} = \{x \in H_i : x \in H \cap G_{i+1}\}$$
$$= H_{i+1}$$

so, $\left({}^{H_i}/_{H_{i+1}}, \otimes\right) \cong (f_i(H_i), \otimes)$

$f_i(H_i) \subseteq {}^{G_i}/_{G_{i+1}}$ and ${}^{G_i}/_{G_{i+1}}$ is a commutative

Hence, $f_i(H_i)$ is a commutative

Therefore, ${}^{H_i}/_{H_{i+1}}$ is a commutative

So, $(H,*)$ is a solvable ■

**<u>Theorem(4-6):</u>**

Let $H\Delta G$ and $G$ is a solvable, then ${}^{G}/_{H}$ is a solvable.