Modules Theory

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المرحلة الرابعة

Definition: A **right module over a ring** R (or **right** R**-module**) is an Additive Abelian group R together with a map $M \times R \to M$ such that to every pair (m, r), where $m \in M, r \in R$, there corresponds a uniquely determined element $mr \in M$ and the following conditions are satisfied:

1. $m(r1 + r2) = mr1 + mr2$
2. $(m1 + m2)r = m1r + m2r$
3. $m(r1r2) = (mr1)r2$
4. $m \cdot 1 = m$

for any $m,m1,m2 \in M$ and any $r, r1, r2 \in R$.

In a similar way one can define the notion of a **left** R**-module**. We shall sometimes write $M = MR$ to emphasize the right action of R. If R is a commutative ring and $M = MR$ then we can make M into a left R-module by defining $rm = mr$ for $m \in M$ and $r \in R$. Thus for commutative rings we can write the ring elements on either side. If R is not commutative, in general not every right R-module is also a left R-module.

In what follows, by saying an R-module we shall mean a right R-module.

Example : Let $M = A$ and as the map $\phi : M \times A \to M$ we take the usual multiplication,

$$\text{i.e., } \phi(m, a) = ma \in M.$$

Example: If M is a vector space over the field R, then M is an R-module.

Example: Let $M = Mmn(R)$ be the set of all $m \times n$ matrices with entries in R. Then M is an R-module, where addition is ordinary matrix addition, and multiplication of the scalar c by the matrix A means multiplication of each entry of A by c.

Example : Let $A = \mathbf{Z}$ be the ring of integers. Then any Abelian group G is a $\mathbf{Z}$-module, if we define the map $\phi : G \times \mathbf{Z} \to G$ as the usual multiplex addition $\phi(g, n) = gn = g + ... + g \in G$.

Definition: A **homomorphism** of a right A-module M into a right A-module N is a map $f : M \to N$ satisfying the following conditions
1. $f(m1 + m2) = f(m1) + f(m2)$ for all $m1,m2 \in M$;
2. $f(ma) = f(m)a$ for all $m \in M$, $a \in A$.

The set of all such homomorphisms f is denoted by $HomA(M,N)$.
If $f, g \in HomA(M,N)$ then $f + g : M \to N$ is defined by $(f + g)(m) = f(m) + g(m)$ for all $m \in M$. One can verify that $f + g$ is also a homomorphism and the set $HomA(M,N)$ forms an additive Abelian group.
If a homomorphism $f : M \to N$ is injective, i.e., $m1 = m2$ implies $f(m1) = f(m2)$, then it is called a **monomorphism**. In order to verify that f is a monomorphism of A-modules it is sufficient to show that $f(m) = 0$ implies $m = 0$.

If a homomorphism $f : M \to N$ is surjective, i.e., every element of N is of the form $f(m)$, then f is called an **epimorphism**.

If a homomorphism $f : M \to N$ is bijective, i.e., injective and surjective, then it is called an **isomorphism** of modules. In this case we say that M and N are **isomorphic** and we shall write M _ N. Isomorphic modules have the same properties and they can be identified. It is easy to check that then $f-1 : N \to M$, defined by $f-1(n) = m$ if and only if $f(m) = n$ is also a homomorphism of modules, so that a bijective homomorphism is an isomorphism in the categorical sense.

Definition: A nonempty subset N of an A-module M is called an A-**submodule** if N is a subgroup of the additive group of M which is closed under multiplication by elements of A. Note that since A itself is a right A-module, submodules of the regular module AA are precisely the right ideals of A.

Let N be a submodule of an A-module M. We say that two elements $x, y \in M$ are equivalent if $x - y \in N$. Consider the set M/N of equivalence classes $m + N$, where $m \in M$. We can introduce a module structure on M/N if we define the operations of addition and multiplication by an element $a \in A$ by setting

$(m + N) + (m1 + N) = (m + m1) + N$,
$(m + N)a = ma + N$ for all $m, m1 \in M$.

Definition: The A-module M/N is called the **quotient module** of M by N. Note that the quotient module has a natural map $\pi : M \rightarrow M/N$ assigning to each element $m \in M$ the class $m + N \in M/N$. Moreover, it is easy to see that π is an epimorphism of A-modules. This epimorphism is called the **natural**.

Homework:

1- Every abelian group A is a Z-module.