**Groups Theory**

**References:**

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* Groups and Numbers, by R. M. Luther.
* A First Course in Abstract Algebra, by J. B. Fraleigh.
* Group Theory, by M. Suzuki.
* Abstract Algebra Theory and Applications, by Thomas W. Judson.
* Abstract Algebra, by I. N. Herstein.
* Basic Abstract Algebra, by P. B. Bhattacharya, S. K. Jain and S. R. Nagpaul.
1. **Definition and Examples of Groups.**

**Definition(1-1):**

A set $G$ is a group if it is satisfying the following four axioms

1. $∃$ a binary operation $G×G⟼G$ (**closure)** $(a,b)⟼ab$
2. $a\left(bc\right)=\left(ab\right)c ∀a,b,c\in G$ (**associativity**),
3. $∃1\in G$ s.t. $a1=a=1a ∀a\in G$
4. $∀a\in G, ∃a^{-1}\in G$ s.t. $aa^{-1}=1=a^{-1}a$ (**inverse**)

**Examples(1-2):**

1. $(R^{\*}=R∖\left\{0\right\}, ∙)$ is a group.

**Solution:** $∀a,b,c\in R^{\*}$, we have

$i. ab\in R^{\*}$, ii. $a\left(bc\right)=\left(ab\right)c,$ iii. $∃1\in R^{\*}\ni a1=a=1a$, iv. $∀a\in R^{\*}, ∃a^{-1}=\frac{1}{a}\in R^{\*}\ni aa^{-1}=1=a^{-1}a$

$2. (Q^{\*}=Q∖\left\{0\right\}, ∙)$ is a group.

$3. (C^{\*}=C∖\left\{0\right\}, ∙)$ is a group.

**Solution:** i, ii are clear,

iii. $∃1\in C^{\*}\ni \left(a+ib\right)1=a+ib=1(a+ib)$,

 iv. $(a+ib)^{-1}=\frac{a-ib}{a^{2}+b^{2}}$

$4. (GL(2,R), ∙)$ is a group.

**Solution:** i, ii are clear, iii.$ ∃\left(\begin{matrix}1&0\\0&1\end{matrix}\right)\in GL\left(2,R\right)\ni \left(\begin{matrix}1&0\\0&1\end{matrix}\right)\left(\begin{matrix}a&b\\c&d\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)=\left(\begin{matrix}a&b\\c&d\end{matrix}\right)\left(\begin{matrix}1&0\\0&1\end{matrix}\right)$, iv. $\left(\begin{matrix}a&b\\c&d\end{matrix}\right)^{-1}=\left(\begin{matrix}\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{matrix}\right)$

5. $(S\_{3}, ∘)$ is a group.

**Solution:** $S\_{3}=\{i, \left(12\right),\left(13\right),\left(23\right),\left(123\right),\left(132\right)\}$

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| $$∘$$ | $$i$$ | $$\left(12\right)$$ | $$\left(13\right)$$ | $$\left(23\right)$$ | $$\left(123\right)$$ | $$\left(132\right)$$ |
| $$i$$ | $$i$$ | $$\left(12\right)$$ | $$\left(13\right)$$ | $$\left(23\right)$$ | $$\left(123\right)$$ | $$\left(132\right)$$ |
| $$\left(12\right)$$ | $$\left(12\right)$$ | $$i$$ | $$\left(132\right)$$ | $$\left(123\right)$$ | $$\left(23\right)$$ | $$\left(13\right)$$ |
| $$\left(13\right)$$ | ? | ? | ? | ? | ? | ? |
| $$\left(23\right)$$ | ? | ? | ? | ? | ? | ? |
| $$\left(123\right)$$ | ? | ? | ? | ? | ? | ? |
| $$\left(132\right)$$ | ? | ? | ? | ? | ? | ? |

We note that axioms i, ii and iii from above table are satisfy axiom iv.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| $$a$$ | $$i$$ | $$\left(12\right)$$ | $$\left(13\right)$$ | $$\left(23\right)$$ | $$\left(123\right)$$ | $$\left(123\right)$$ |
| $$a^{-1}$$ | ? | ? | ? | ? | ? | ? |

6. $(G=\left\{0,-1,1,2\right\},+)$ is not a group.

**Solution**: since $1+2=3\notin G$

7. $(G=\left\{-1,1\right\},∙)$ is a group.

**Solution**:

|  |  |  |
| --- | --- | --- |
| $$∙$$ | $$-1$$ | $$1$$ |
| $$-1$$ | ? | ? |
| $$1$$ | ? | ? |

8. Let $G=\{a,b,c,d\}$ be a set. Define a binary operation $\*$ on $G$ by the following table

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$\*$$ | $$a$$ | $$b$$ | $$c$$ | $$d$$ |
| $$a$$ | $$a$$ | $$b$$ | $$c$$ | $$d$$ |
| $$b$$ | $$b$$ | $$c$$ | $$d$$ | $$a$$ |
| $$c$$ | $$c$$ | $$d$$ | $$a$$ | $$b$$ |
| $$d$$ | $$d$$ | $$a$$ | $$b$$ | $$c$$ |

Show that $(G,\*)$ is a group.

**Solution**: axioms i,ii are satisfy from above table, iii. The identity element is $a$, axiom iv.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$x$$ | $$a$$ | $$b$$ | $$c$$ | $$d$$ |
| $$x^{-1}$$ | ? | ? | ? | ? |

9. $(G=\left\{1,-1,i,-i\right\},∙)$ is a group.

**Solution**:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$∙$$ | $$1$$ | $$-1$$ | $$i$$ | $$-i$$ |
| $$1$$ | ? | ? | ? | ? |
| $$-1$$ | ? | ? | ? | ? |
| $$i$$ | ? | ? | ? | ? |
| $$-i$$ | ? | ? | ? | ? |

10. Let $G=Z, a\*b=a+b+2$, show that $(G,\*)$ is a group.

**Solution**:$ ∀a,b,c\in Z$, we have i. $a\*b=a+b+2\in Z$,

 ii. $a\*\left(b\*c\right)=a\*\left(b+c+2\right)=a+b+c+4, \left(a\*b\right)\*c=\left(a+b+2\right)\*c=a+b+c+4$,

 iii. $a\*u=a+u+2=a, u=-2,$

iv. $a\*z=-2⟹a+z+2=-2⟹z=-a-4$

11. Let $G=\{f\_{1}, f\_{2}, f\_{3}, f\_{4}\}$ with $f\_{i}$ s.t. $i=1,2,3,4$ are mappings on $R∖\{0\}$ s.t. $f\_{1}\left(x\right)=x, f\_{2}\left(x\right)=-x, f\_{3}\left(x\right)=\frac{1}{x}, f\_{4}\left(x\right)=-\frac{1}{x}$. Show that$(G,∘)$ is a group.

**Solution**:

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$∘$$ | $$f\_{1}$$ | $$f\_{2}$$ | $$f\_{3}$$ | $$f\_{4}$$ |
| $$f\_{1}$$ | ? | ? | ? | ? |
| $$f\_{2}$$ | ? | ? | ? | ? |
| $$f\_{3}$$ | ? | ? | ? | ? |
| $$f\_{4}$$ | ? | ? | ? | ? |

12. Let$ G=R×R=\{\left(a,b\right):a,b\in R, a\ne 0\}$ and $\*$ be defined by $\left(a,b\right)\*\left(c,d\right)=(ac,bc+d)$. Show that $(G,\*)$ is a group.

**Solution**: i. $\left(a,b\right)\*\left(c,d\right)=(ac,bc+d)\in G$

ii.$ \left(a,b\right)\*\left[\left(c,d\right)\*\left(e,f\right)\right]=\left(a,b\right)\*\left(ce,de+f\right)=\left(ace,bce+de+f\right), \left[\left(a,b\right)\*\left(c,d\right)\right]\*\left(e,f\right)=\left(ac,bc+d\right)\*\left(e,f\right)=(ace,bce+de+f)$,

 iii. $\left(a,b\right)\*\left(x,y\right)=\left(a,b\right)⟹\left(ax,bx+y\right)=\left(a,b\right)⟹x=1, bx+y=b⟹b+y=b⟹y=0$,

 iv.$ \left(a,b\right)\*\left(w,z\right)=\left(1,0\right)⟹\left(aw,bw+z\right)=\left(1,0\right)⟹w=\frac{1}{a}, ba^{-1}+z=0⟹z=\frac{-b}{a}$

13. Let $(G,\*)$ be an arbitrary group, the set of the functions from $G$ into $G$ with the composition $(F\_{G},∘)$ is forms a group, where $F\_{G}=\left\{f\_{a}:a\in G\right\},f\_{a}:G⟼G$ s.t. $f\_{a}\left(x\right)=a\*x,x\in G$.

**Solution:** i. Let $f\_{a, }f\_{b}\in F\_{G},a,b\in G$

$$\left(f\_{a° }f\_{b}\right)\left(x\right)=f\_{a }\left(f\_{b}\left(x\right)\right)=f\_{a}\left(b\*x\right)=a\*\left(b\*x\right)=\left(a\*b\right)\*x=f\_{a\*b}(x)\in F\_{G}$$

ii. $\left(f\_{a° }f\_{b}\right)∘f\_{c}=f\_{a\*b}∘f\_{c}=f\_{\left(a\*b\right)\*c}=f\_{a\*\left(b\*c\right)}=f\_{a}∘f\_{b\*c}=f\_{a}∘(f\_{b}∘f\_{c})$

iii.$ f\_{e}$ is an identity of $F\_{G}$, since $f\_{a}∘f\_{e}=f\_{a\*e}=f\_{e\*a}=f\_{e}∘f\_{a}=f\_{a}$

iv. the inverse of $f\_{a}$ in $F\_{G}$ is $f\_{a^{-1}}$, since $f\_{a}∘f\_{a^{-1}}=f\_{a\*a^{-1}=}f\_{a^{-1}\*a}=f\_{a^{-1}}∘f\_{a}=f\_{e}$

14. Let $n$ be a positive integer and take $w=\cos((\frac{2π}{n}))+i\sin((\frac{2π}{n})\in C)$, then $(C\_{n}=\left\{1,w,w^{2},…, w^{n-1}\right\},∙)$ is an abelian group.

**Definition(1-3):**  A group $(G,\*)$ is an abelian if $a\*b=b\*a ∀a,b\in G$.

**Example(1-4):** Determine whether the previous examples are abelian .

**Exercises:**

1. Determine whether $(G,\*)$ an abelian group.
* $G=Z, a\*b=a+b+3$
* $G=R×R=\{\left(a,b\right):a,b\in R\}$ s.t. $\left(a,b\right)\*\left(c,d\right)=(a+b,b+d+2bd)$
* $(G=\left\{f\_{1}, f\_{2}, f\_{3}, f\_{4},f\_{5},f\_{6}\right\},∘)$ where $f\_{1}\left(x\right)=x, f\_{2}\left(x\right)=\frac{1}{x},f\_{3}\left(x\right)=1-x, f\_{4}\left(x\right)=\frac{x-1}{x}, f\_{5}\left(x\right)=\frac{x}{x-1}, f\_{6}\left(x\right)=\frac{1}{1-x}$
* $G=\{\left(a,b\right):a,b\in R, a\ne 0,b\ne 0\}$ s.t. $\left(a,b\right)\*\left(c,d\right)=(ab,bd)$
* $(G=\left\{an:n\in Z\right\},+)$
* $G=Q^{\*},a\*b=\frac{ab}{2}$
1. Show that, $(G=\left\{\left(\begin{matrix}1&0\\0&1\end{matrix}\right),\left(\begin{matrix}0&1\\-1&0\end{matrix}\right) , \left(\begin{matrix}-1&0\\0&-1\end{matrix}\right),\left(\begin{matrix}0&-1\\1&0\end{matrix}\right)\right\},∙)$ is a group.
2. Show that, $(C\_{8},∙)$ is an abelian group.