**5. Subgroups and Their Properties**

**Definition(5-1):** Let $(G,\*)$ be a group and $H⊂G$, $H$ a non-empty subset of $G$. Then $(H,\*)$ is a subgroup of $(G,\*)$, if $(H,\*)$ is itself a group.

 **Definition(5-2):** Let $(G,\*)$ be a group and $H⊂G$, then $(H,\*)$ is a subgroup of $(G,\*)$ if,

1. $∀ a, b \in H⟹a\*b\in H$;
2. The identity element of $G$ is an element of $H$, $(e\in G⟹e\in H)$;
3. $∀ a\in H⟹a^{-1}\in H$.

**Remark(5-3):** Each group $(G,\*)$ has at least two subgroups$ (\{e\},\*)$ and $(G,\*)$, these subgroups are known trivial subgroups and improper, any subgroup different from these subgroups known proper subgroup.

**Example(5-4):** $(Z,+)$ is a proper subgroup of $(R,+)$.

**Example(5-5):** $(H=\{-1,1\},∙)$ is a proper subgroup of $(G=\{-1,1,-i,1\},∙)$.

**Example(5-6):** $(H=\{0,2\},+\_{4})$ is a proper subgroup of $(Z\_{4},+\_{4})$, but $(H=\{0,3\},+\_{4})$ not subgroup of $(Z\_{4},+\_{4})$.

**Example(5-7):** $(Q∖\{0\},∙)$ is a subgroup of $(R∖\{0\},∙)$.

**Theorem(5-8):** Let $(G,\*)$ be a group and $H⊂G$, then $(H,\*)$ is a subgroup of $(G,\*)$ iff $a\*b^{-1}\in H, ∀ a,b \in H$.

**Proof:** $(⟹)$ let $(H,\*)$ be a subgroup of $(G,\*)$ and $a,b\in H$, then $a,b^{-1}\in H⟹a\*b^{-1}\in H$

$(⟸)$ let $a\*b^{-1}\in H$, to prove $(H,\*)$ be a subgroup of $(G,\*)$

1. Since $H\ne ∅⟹∃ b\in H \ni b\*b^{-1}\in H⟹e\in H$;
2. Since $b\in H$ and $e\in H⟹e\*b^{-1}\in H⟹b^{-1}\in H$;
3. Let $a\in H$ and $b^{-1}\in H⟹a\*\left(b^{-1}\right)^{-1}\in H⟹a\*b\in H⟹(H,\*)$ is a subgroup of $(G,\*)$.

**Example(5-9):** Let $(Z,+)$ be a group and $H=\{5a:a\in Z\}$. Show that $(H,+)$ is a subgroup of $(Z,+)$.

**Solution:** let $x,y\in H$, to prove $x+y^{-1}\in H$

$$x\in H⟹x=5a, a\in Z$$

$$y\in H⟹y=5b, b\in Z$$

$$x+y^{-1}=5a+(5b)^{-1}=5a+5\left(-b\right)=5(a-b)\in H$$

$⟹(H,+)$ is a subgroup of $(Z,+)$.

**Theorem(5-10):** If $(H\_{i},\*)$ is the collection of subgroup of $(G,\*)$, then $(∩H\_{i},\*)$ is also subgroup of $(G,\*)$.

**Proof:** 1. Since $∃e\in H\_{i}, ∀i⟹e\in ∩H\_{i}⟹∩H\_{i}\ne ∅$;

2. let $x,y\in ∩H\_{i}$, to prove $x\*y^{-1}\in ∩H\_{i}$

Since $x,y\in ∩H\_{i}⟹x,y\in H\_{i}, ∀i⟹x\*y^{-1}\in H\_{i}, ∀i$

$⟹x\*y^{-1}\in ∩H\_{i}⟹(∩H\_{i},\*)$ is a subgroup of $(G,\*)$.

**Theorem(5-11):** Let $(H\_{i},\*)$ be the collection of subgroups of $(G,\*)$ and let$ H\_{k}$ and $ H\_{j}\in \{H\_{i}\}$ such that there is $ H\_{e}\in \left\{H\_{i}\right\}, H\_{k}⊆ H\_{l} $and $ H\_{j}⊆ H\_{l}$, then $(⋃H\_{i},\*)$ is also subgroup of $(G,\*)$.

**Proof:** 1. Since $∃e\in H\_{i}$ for some$i$ $⟹e\in ⋃H\_{i}⟹⋃H\_{i}\ne ∅$;

2**.** let$ x,y\in ⋃H\_{i}$, then $x,y\in H\_{k}$ or $x,y\in H\_{j}$, so $x,y\in H\_{l} $

$$⟹x\*y^{-1}\in H\_{l}⟹x\*y^{-1}\in ⋃H\_{i}$$

$⟹(⋃H\_{i},\*)$ is a subgroup of $(G,\*)$.

**Theorem(5-12):** Let$(H\_{1},\*)$ and $(H\_{2},\*)$ are two subgroups of $(G,\*)$, then $(H\_{1}⋃H\_{2},\*)$ is a subgroup of $(G,\*)$ iff $H\_{1}⊂H\_{2}$ or $H\_{2}⊂H\_{1}$.

**Proof:** $(⟹)$ let $(H\_{1}⋃H\_{2},\*)$ is a subgroup of $(G,\*)$,

to prove $H\_{1}⊂H\_{2}$ or $H\_{2}⊂H\_{1}$

suppose that $H\_{1}⊄H\_{2}$ and $H\_{2}⊄H\_{1}$

$⟹∃a\in H\_{1}, a\notin H\_{2}$ and $∃b\in H\_{2}, b\notin H\_{1}$

$$⟹a,b\in H\_{1}⋃H\_{2}⟹a\*b^{-1}\in H\_{1}⋃H\_{2}$$

$⟹a\*b^{-1}\in H\_{1}$ or $a\*b^{-1}\in H\_{2}$

$⟹a,b\in H\_{1}$ or $a,b\in H\_{2}$, but this is contradiction

$⟹$ $H\_{1}⊂H\_{2}$ or $H\_{2}⊂H\_{1}$

$(⟸)$ let $H\_{1}⊂H\_{2}$ or $H\_{2}⊂H\_{1}$

To prove $(H\_{1}⋃H\_{2},\*)$ is a subgroup of $(G,\*)$

If $H\_{1}⊂H\_{2}⟹H\_{1}⋃H\_{2}=H\_{2}$ is a subgroup of $(G,\*)$

If $H\_{2}⊂H\_{1}⟹H\_{1}⋃H\_{2}=H\_{1}$ is a subgroup of $(G,\*)$

$⟹(H\_{1}⋃H\_{2},\*)$ is a subgroup of $(G,\*)$.

**Remark(5-13):** $(H\_{1}⋃H\_{2},\*)$ need not be a subgroup of $(G,\*)$, for example:

$H\_{1}=\{r\_{1},r\_{3}\}$ is a subgroup of $G\_{S}$

$H\_{2}=\{r\_{1},v\}$ is a subgroup of $G\_{S}$

$H\_{1}⋃H\_{2}=\{r\_{1},r\_{3},v\}$ is not a subgroup of $G\_{S}$, since $r\_{3}∘v=h\notin H\_{1}⋃H\_{2}$.

**Definition(5-14):** Let $(G,\*)$ be a group and $\left(H,\*\right), (K,\*)$ are two subgroups of $(G,\*)$, then the product of $H$ and $K$ is the set:

$$H\*K=\{h\*k:h\in H, k\in K\}$$

**Notes(5-15):**

1. $H\*H$ is write $H^{2}$;
2. If $H=\{a\}$, then $H\*K=a\*K$. If $K=\{b\}$, then $H\*K=H\*b$;
3. $H⋃K⊆H\*K$.

**Theorem(5-16):** Let $(G,\*)$ be a group and $\left(H,\*\right), (K,\*)$ are two subgroups of $(G,\*)$, then

1. $H\*K\ne ∅$ and $H\*K⊆G$.
2. $H⊆H\*K$ and $K⊆H\*K$.
3. $(H\*K,\*)$ is a subgroup of $(G,\*)$ iff $ H\*K=K\*H$.
4. If $(G,\*)$ is an abelian group, then $(H\*K,\*)$ is a subgroup of$ (G,\*)$.

**Proof:**

1. $e\in H$ and $e\in K⟹e\*e=e\in H\*K⟹H\*K\ne ∅$, and let $x\in H\*K⟹x=a\*b\ni a\in H⊆G$, and $b\in K⊆G⟹a\in G$, and $b\in G⟹a\*b=x\in G⟹H\*K⊂G$.
2. Let$ x\in H⟹x=x\*e\in H\*K⟹x\in H\*K⟹H⊆H\*K$, similarly, $K⊆H\*K$.
3. $(⟹)$ suppose$ (H\*K,\*)$ is a subgroup of $(G,\*)$, to prove $H\*K=K\*H$, this means $H\*K⊆K\*H$ and $K\*H⊆H\*K$, let $x\in H\*K⟹x=a\*b\ni a\in H$ and $b\in K$, since $H\*K$ is a subgroup of $G⟹x^{-1}\in H\*K$, let $x^{-1}=c\*d\ni c\in H$ and$ d\in K$, $x=(x^{-1})^{-1}=(c\*d)^{-1}=d^{-1}\*c^{-1}\ni d^{-1}\in K$ and$ c^{-1}\in H⟹x=d^{-1}\*c^{-1}\in K\*H⟹H\*K⊆K\*H,$ to prove $K\*H⊆H\*K$ (**Homework)**.

$ (⟸)$ let$H\*K=K\*H$, to prove $(H\*K,\*)$ is a subgroup of $(G,\*)$

 $H\*K\ne ∅$ and $H\*K⊆G$ (by 1)

Let $x,y\in H\*K$, to prove $x\*y^{-1}\in H\*K$

$x\in H\*K⟹x=a\*b\ni a\in H$ and $b\in H$

$y\in H\*K⟹y=c\*d\ni c\in H$ and $d\in H$

$$x\*y^{-1}=\left(a\*b\right)\*(c\*d)^{-1}$$

 $=\left(a\*b\right)\*(d^{-1}\*c^{-1})$

 $=a\*\left(b\*d^{-1}\right)\*c^{-1}$

$$⟹\left(b\*d^{-1}\right)\*c^{-1}\in K\*H=H\*K$$

$$⟹\left(b\*d^{-1}\right)\*c^{-1}\in H\*K$$

$$⟹∃p\in H, q\in K\ni \left(b\*d^{-1}\right)\*c^{-1}=p\*q$$

$$⟹a\*\left(b\*d^{-1}\right)\*c^{-1}=a+p+q\in H\*K$$

$$⟹x\*y^{-1}\in H\*K$$

$⟹(H\*K,\*)$ is a subgroup of $(G,\*)$.

1. $H\*K\ne ∅$, let $x,y\in H\*K$

To prove $x\*y^{-1}\in H\*K$

$x\in H\*K⟹x=a\*b\ni a\in H$ and $b\in K$

$y\in H\*K⟹y=c\*d\ni c\in H$ and $d\in K$

$$x\*y^{-1}=\left(a\*b\right)\*(c\*d)^{-1}$$

$ $ $=\left(a\*b\right)\*(d^{-1}\*c^{-1})$

 $=\left(a\*b\right)\*(c^{-1}\*d^{-1})$

 $=a\*\left(b\*c^{-1}\right)\*d^{-1}$

 $=(a\*c^{-1})\*(b\*d^{-1})$

$$⟹x\*y^{-1}\in H\*K$$

$⟹(H\*K,\*)$ is a subgroup of $(G,\*)$.

**Example(5-17):** In $(Z\_{8},+\_{8})$, let $H=\{0,4\}$ and $K=\{0,2,4,6\}$. Find $H+\_{8}K$.

**Solution:** $H+\_{8}K=\{0,2,4,6\}$.

**Note(5-18):** Let $(H,\*)$ and $(K,\*)$ are two subgroups of$ (G,\*)$, then:

1. $H\*K\ne K\*H$;
2. $(H\*K,\*)$ need not be a subgroup of$ (G,\*)$, give example (**Homework).**

**Example(5-18):** Is $H=\{0,6\}$ is a subgroup of $(Z\_{8},+\_{8})$? (**Homework).**

**Example(5-19):** Is $H=\{0,12\}$ is a subgroup of $(Z\_{4},+\_{4})$? (**Homework).**

**Definition(5-20):** The center of a group $(G,\*)$ denoted by Cent($G$) or $C(G)$ is the set $C\left(G\right)=\{c\in G:c\*x=x\*c, ∀ x\in G\}$.

**Note(5-21):** $C(G)\ne ∅$, since $∃e\in G\ni e\*x=x\*e ∀x\in G⟹e\in C(G)$.

**Example(5-22):** The group $(R∖\left\{0\right\},∙)$, $C\left(R\right)=R$, since $(R∖\left\{0\right\},∙)$ is an abelian group.

**Example(5-23):** The group $(S\_{3},∘)$, $C\left(S\_{3}\right)=\{f\_{1}\}$, since

 $C\left(S\_{3}\right)=\left\{f\in S\_{3}:f∘g=g∘f ∀g\in S\_{3}\right\}=\{f\_{1}\}$.

**Theorem(5-24):** Let $(G,\*)$ be a group. Then$(C(G),\*)$ is a subgroup of $(G,\*)$.

**Proof:** $C(G)\ne ∅$, $C\left(G\right)=\{a\in G:x\*a=a\*x, ∀ x\in G\}⊆G$

let $a,b\in C(G)$, to prove $a\*b^{-1}\in C(G)$

$$a\in C\left(G\right)⟹a\*x=x\*a ∀ x\in G$$

$$b\in C\left(G\right)⟹b\*x=x\*b ∀ x\in G$$

To prove $\left(a\*b^{-1}\right)\*x=x\*\left(a\*b^{-1}\right) ∀x\in G$

$$\left(a\*b^{-1}\right)\*x=a\*(b^{-1}\*x)$$

 $=a\*(x^{-1}\*b)^{-1}$

 $=a\*(b\*x^{-1})^{-1}$

 $=a\*(x\*b^{-1})$

 $=(a\*x)\*b^{-1}$

 $=(x\*a)\*b^{-1}$

 $=x\*(a\*b^{-1})$

$$⟹\left(a\*b^{-1}\right)\in C(G)$$

$⟹(C\left(G\right),\*)$ is a subgroup of$ (G,\*)$.

**Theorem(5-25):** Let $ (G,\*)$ be a group, then $C\left(G\right)=G$ iff $G$ is an abelian group.

**Proof:** $\left(⟹\right) ∀ a\in G⟹a\in C(G)$

$$⟹a\*x=x\*a ∀ x\in G$$

$$⟹a\*x=x\*a ∀ x,a\in G$$

$⟹G$ is an abelian group.

$\left(⟸\right)$ suppose that $G$ is an abelian group, to prove $C\left(G\right)=G$

This means $C(G)⊆G$ and $G⊆C(G)$

By definition of $C\left(G\right), C(G)⊆G$

To prove $G⊆C(G)$

Let $x\in G, G$ is an abelian group

$$⟹x\*a=a\*x ∀a\in G$$

$$⟹x\in C(G)$$

$$⟹G⊆C(G)$$

$⟹C\left(G\right)=G$.