**6. More Results of Subgroups**

**Cyclic Group:**

**Definition(6-1)**Let be a group and , the cyclic subgroup of generated by is denoted by and defined as

If , then is called a cyclic group.

**Definition(6-2):** A group is called cyclic group generated by iff.

**Example(6-3):** In , find the cyclic subgroup generated by .

**Solution:**

is a cyclic group generated by.

is a cyclic subgroup of .

is generated by .

**Example(6-4):** In , find a cyclic group generated by .

**Solution:**

is a cyclic group generated by and .

**Example(6-5):** Is a cyclic group?

**Solution:**

is not a cyclic group.

**Example(6-6):** In , find a cyclic subgroup generated by . (**Homework**)

**Theorem(6-7):** Every cyclic group is an abelian.

**Proof:** let be a cyclic group,

To prove is an abelian group

Let , to prove

is an abelian group.

**Note(6-8):** The converse of above theorem is not true in general, for example.

is an abelian group, but is not a cyclic group, since

is not a cyclic.

**Theorem(6-9):** .

**Proof:**

.

**Theorem(6-10):** If is a finite group of order generated by, then , such that is the least positive integer , this means .

**Example(6-11):** Show that is a cyclic group.

**Solution:**

, to prove

and .

**Definition(6-12):** (Division Algorithm for )

If are integers, with . Then there is a unique pair of integers.

The number is called the quotient and is called the remainder when is divided by .

**Example(6-13):** Find the quotient and remainder, when is divided by according to the division algorithm.

**Solution:**

.

**Example(6-14):** .

**Solution:**

.

**Example(6-15):** .

**Solution:**

.

**Theorem(6-16):** A subgroup of a cyclic group is a cyclic.

**Proof:** let be a cyclic group generated by and let be a subgroup of

If , then is a cyclic

If and ( is a proper subgroup), then

Let be a least positive integer such that

to prove

to prove

let

by division algorithm of and

but

To prove

Let

is a cyclic subgroup.

**Corollary(6-17):** If is a finite cyclic group of order generated by, then every subgroup of is a cyclic generated by .

**Proof:** suppose is a finite,

Let be a subgroup of , then is a cyclic

such that , to prove

, by division algorithm of

, but

If .

**Example(6-18):** Find all subgroups of .

**Solution:** ,

If

If

If

If .

**Corollary(6-19):** If is a finite cyclic group of prime order, then has no a proper subgroup.

**Proof:** let be a finite group such that

( is a prime number)

Let be a cyclic subgroup

or

If (not a proper subgroup)

If (not a proper subgroup)

has no a proper subgroup.

**Example(6-20):** Find all subgroup of .

**Solution:**

Let or

If

If .

**Definition(6-21):** A positive integer is said to be a greatest common divisor of two non-zero numbers iff

1. If .

**Example(6-22):** Find g. c. d..

**Solution:** g. c. d.**,** since

.

**Remark(6-23):** If is a finite cyclic group of order generated by , then the generator of is .

**Example(6-24):** Find all generators of .

**Solution:**

therefore, the generators of are .

**Theorem(6-25):** If is an infinite cyclic group generated by , then:

1. The numbers are only generators of ;
2. Every subgroup of except is an infinite subgroup.

**Proof:** (1) suppose , to prove

Let

Let

Substitute in , we get

or

If

If .

(2) let be a subgroup of

To prove is an infinite

Suppose that is a finite such that

is a cyclic subgroup

, but this is contradiction

is a finite)

Thus, is an infinite.

**Definition(6-26):** Let be a subgroup of a group . The set of is the left coset of containing , while the subset is the right coset of containing .

**Example(6-27):** If , then

**Notes(6-28):**

1. is not subgroup ( in general), give an example (**Homework**);
2. (in general), for example

.

**Theorem(6-29):** Let be a subgroup of and , then

1. is itself left coset of in .

**Proof:** .

1. If is an abelian group, then .

**Proof:** *.*

The converse of above theorem is not true in general, for example

, but is not an abelian.

**Proof:** .

1. iff

**Proof:**  suppose that, then by .

suppose that , to prove

This means and

Let

To prove

Let

.

1. iff

**Proof:**

, by

suppose that

by .

1. or

**Proof:** suppose that

To prove

and

and

by

or suppose

to prove

suppose

and

and

, but this is contradiction

.

1. The set of all distinct left coset of in form a partition on.

**Proof:** to prove and

are distinct

To prove

(by definition of a coset)

From , we have .

**Note(6-30):** Every coset (left or right) of a subgroup of a group has the same number of elements as .

**Example(6-31):** The group is an abelian. Find the partition of into coset of the subgroup .

**Solution:**

All the cosets of are and since is an abelian group, then the left coset is an equal to the right coset.

**Example(6-32):** In , let . Find the partition of into left coset of and the partition into right coset of . (**Homework**)

**Definition(6-33):** Let be a subgroup of a group . The number of left cosets or right cosets of in is called the index of in and denoted by.

**Note(6-34):** If is a finite group, then .

**Example(6-35):**

**Example(6-36):**

**Theorem(6-37):** (Lagrange Theorem)

Let be a subgroup of a finite group . Then the order of is a divisor of the order of .

**Proof:** let be a finite group and be a subgroup of

To prove (to prove)

Since is a finite

Let are left cosets of

and

-times

**Corollary(6-38):** If is a finite group, then the order of any element of divides the order of.

**Proof:** suppose that is a finite such that

Let has a finite order such that

To prove such that

Since is a cyclic group

(by Lagrange Theorem)

**Corollary(6-39):** If is a finite group, then .

**Proof:** suppose that

Let (by Corollary of Lagrange)

.

**Corollary(6-40):** Every group of prime order is a cyclic.

**Proof:** let be a finite

or

If

If

is a cyclic group.

**Corollary(6-41):** Every group of order less than is an abelian.

**Proof:** let be a finite group

or or or or

If is an abelian

If or or is a cyclic is an abelian

If or or

If

If

is an abelian

If

is a cyclic is an abelian.