**7. Normal Subgroups and Quotient Groups**

**Definition(7-1):**Let be a group and , then is a conjugate to and denoted by iff and iff .

iff

**Example(7-2):** In , is ?

**Solution:**

Is and ? (**Homework**)

**Example(7-3):** In , is ?

**Solution:**

**Remark(7-4):** If is an abelian group and , then .

**Proof:** suppose that

**Theorem(7-5):** The relation (conjugate) is an equivalent relation.

**Proof:** (1) reflexive

let , to prove

(2) symmetric

Let and , to prove

(3) transitive

Let and , to prove

Substitute in , we get

( where )

.

**Definition(7-6):** Let be a group and , then the conjugate of is denoted by and defined as

or

or

The set of all elements conjugate to is called the conjugate class of .

**Examples(7-7):** Find the conjugate class of each element in the following groups:

1. (**Homework**)
2. (**Homework**)
3. .

**Solution:**

.

**Example(7-8):** Find in .

**Solution:**

(by Remark if is an abelian group and , then )

**Note(7-9):** Let be a group and , then need not be a subgroup of , for example in , is not a subgroup of .

**Theorem(7-10):** Let be a group and , then

1. .

**Proof:** since ( is a reflexive)

1. .

**Proof:**  suppose that , to prove

By

suppose that , to prove

This means and

Let and

Let and

From , we get

1. iff (**Homework**)
2. or (**Homework**)

**Proof:**  let ( by Theorem)

.

1. is an abelian group.

**Proof:**

is an abelian group.

1. (**Homework**)
2. (**Homework**)

**Definition(7-11):** Let be a group and , then the normalizer of is denoted by and defined as .

**Example(7-12):** In . Find .

**Solution:**

**Theorem(7-13):** Let be a group and , then

1. is a subgroup of .

**Proof:**

Since

Closure: let , to prove

Let , to prove

Since

is a subgroup.

1. (**Homework**)
2. is an abelian.

**Proof:** suppose that , to prove is an abelian

is an abelian

suppose that is an abelian, to prove

This means and

(by definition of )

To prove

Let and is an abelian

1. (**Homework**)

**Proof:**

Define

To prove is a map, is an one to one, is an onto (**Homework**)

1. If is a finite group, then

**Proof:** by is a subgroup of

By Lagrange Theorem

**Definition(7-14):** Let are two subgroups of , then is a conjugate subgroup of iff and denoted by .

**Example(7-15):** In . Is ?

**Solution:** this means, ?

.

**Example(7-16):** In . Is ?

**Solution:** this means,

Since

.

**Example(7-17):** In , let . Is ?

(**Homework**)

**Theorem(7-18):** Let are two subgroups of and , then .

**Proof:** since

To prove

Define

To prove is a map ?

Let , to prove

Since

is a map.

Is an one to one ? let

is an one to one.

Is an onto?

is an onto.

.

**Theorem(7-19):** Let be a subgroup of and , then is a subgroup of.

**Proof:**  and

Let , to prove

Let

Let

is a subgroup of .

**Note(7-20):** The relation of conjugate is equivalent relation on the set of all subgroups of . (**Homework**)

**Definition(7-21):** Let be a subgroup of , then the conjugate class of is denoted by and define as

**Example(7-22):**, find .

**Solution:**

**Example(7-23):** , is the four-Klien group. is an abelian, , find .

**Solution:**

.

**Deffinition(7-24):** Let be a subgroup of , then the normalizer of is denoted by and defined as

**Example(7-25):** The group , find .

**Solution:**

**Examples(7-26):** Find to each of the following:

1. The group . (**Homework**)
2. The group . (**Homework**)
3. The group . (**Homework**)

**Theorem(7-27):** Let be a subgroup of , then

1. is a subgroup of containing .

**Proof:** since

Let , to prove

This means

Since

is a subgroup of

To prove

Let

1. If is an abelian group, then .

**Proof:** suppose that is an abelian group, to prove

This means

By definition of

Let

1. (**Homework**)
2. If is a finite group, then

**Note(7-28):** If , then is an abelian group. (**Homework**)

**Definition(7-29):** A subgroup is called a self-conjugate iff , this means .

**Example(7-30):** In

is a self-conjugate

is not a self-conjugate.

**Definition(7-31):** A subgroup is called a normal subgroup of denoted by is a self-conjugate

Or

**Example(7-32):** The group

**Example(7-33):** The group

**Example(7-34):** The group

**Theorem(7-35):** Let be a subgroup of , then

1. .

**Proof:**

**Proof:**  suppose that , to prove

This means

(by definition of )

To prove

Let

suppose that , to prove

(by )

**Proof: :**  suppose that , to prove

Since by definition

suppose that

To prove , this means

Which is

To prove

Let

From , we get

**Proof:**  suppose that

suppose that

, but this is contradiction

**Theorem(7-36):** Let be a group, then

1. (**Homework**)
2. (**Homework**)
3. (**Homework**)

**Theorem(7-37):** Every subgroup of an abelian group is a normal subgroup.

**Proof:** let be an abelian group and be a subgroup of,

to prove

.

**Note(7-38):** The converse of above theorem is not true, for example

is not an abelian.

The subgroups of are

**Theorem(7-39):** Let be a subgroup of , then .

**Proof:** since , then there are two distinct left (right) cosets of in . (left cosets of in )

(right cosets of in )

If

If

.

**Note(7-40):** The converse of above theorem is not true, for example

, but .

**Note(7-41):** If , then , where are two subgroups of the group .

Consider and

, since .

, since

.

**Definition(7-42):** A group is called a simple group iff has no proper normal subgroup.

**Examples(7-43):**

1. The group is not a simple, since .
2. The group is not a simple, since .
3. The group is not a simple, since .
4. The group is a simple group, since has no proper subgroup.

**Definition(7-44):** Let and . Define on as follows:, is called a quotient group of by .

**Theorem(7-45):** Let , then is a group.

**Proof: ,** since

Closure: let ,

Associative: let

Identity:

is an identity element of

Inverse: let , to prove

is a group.

**Example(7-46):** In the group , find (if exist).

**Solution:** exist

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

is a quotient group, is an identity.

**Example(7-47):** In the group , find (if exist).(**Homework**)

**Example(7-48):** In the group , , find (if exist).

**Solution:** since exist

But if is not exist.

**Theorem(7-49):** The quotient group of an abelian is an abelian.

**Proof:** suppose that is an abelian group and is a subgroup of is a group

Let

is an abelian group.

**Theorem(7-50):** If is a cyclic group, then is a cyclic group.

**Proof:** suppose that is a cyclic group, is a subgroup of .

, since is a cyclic is an abelian

is a group. To prove is a cyclic group, this means there is , to prove

, let

-times

-times

To prove , let

Therefore, is a cyclic group.

**Note(7-51):** The converse of above theorem is not true, for example:

, is a group,

(prime order), is a cyclic group, but is not a cyclic

**Theorem(7-52):** Let be a group and is a cyclic group, then is an abelian group.

**Note(7-53):** The converse of this theorem is not true, for example:

, is an abelian (not a cyclic)

is not a cyclic.

**Definition(7-54):** Let be a group. If , then the commutator of is .

The commutator , this means are commute, the identity element is a commutator.

**Example(7-55):** In the group.

**Example(7-56):** In the group.

**Note(7-57):** The commutator is an identity iff is an abelian group.

**Definition(7-58):** Let be a group, then the commutator subgroup of denoted by is the collection of all the finite products of commutators in .

**Theorem(7-59):** The group is a normal subgroup.

**Proof:** to prove is a subgroup of .

, since

Let , to prove

Thus, is a subgroup of .

To prove is a normal subgroup, let

To prove , let

Therefore, is a normal subgroup of .

**Theorem(7-60):** Let be a normal subgroup of , then is an abelian iff .

**Proof:** suppose that and is an abelian

.

**Corollary(7-61):** Prove that is an abelian group. (**Homework**)