1. **The Jordan-Holder Theorem and Related Concepts.**

**Definition(10-1):**

By a *chain* for a group $(G,\*)$ is meant any finite sequence of subsets of

$G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$ descending from $G$ to $\{e\}$ with the property that all the pairs $(H\_{i},\*)$ are subgroups of $(G,\*)$.

 **Remark(10-2):**

The integer $n$ is called the length of the chain. When $n=1$, then the chain in definition (1-1) will called the trivial.

 **Example(10-3):**

 Find all chains in a group $(Ζ\_{4}, +\_{4})$.

 **Solution:** The subgroups of a group $(Ζ\_{4}, +\_{4})$ are :

* $H\_{1}=(Ζ\_{4}, +\_{4})$
* $H\_{2}=(\{0\}, +\_{4})$
* $H\_{3}=\left(\left〈2\right〉,+\_{4}\right)=(\left\{0,2\right\},+\_{4})$

The chains of a group $(Ζ\_{4}, +\_{4})$ are

$Ζ\_{4}⊃\{0\}$ is a chain of length one

$Ζ\_{4}⊃\left〈2\right〉⊃\{0\}$ is a chain of length two.

 **Example(10-4):**

 In the group $(Ζ\_{12}, +\_{12})$ of integers modulo $12$, the following chains are normal chains:

$Ζ\_{12}⊃\left〈6\right〉⊃\{0\}$,

$Ζ\_{12}⊃\left〈2\right〉⊃\left〈4\right〉⊃\{0\}$,

$Ζ\_{12}⊃\left〈3\right〉⊃\left〈6\right〉⊃\{0\}$,

$Ζ\_{12}⊃\left〈2\right〉⊃\left〈6\right〉⊃\{0\}$.

 All subgroups are normal, since $(Ζ\_{12}, +\_{12})$ is a commutative group.

**Definition(10-5):** (***Normal Chain***)

If $(H\_{i},\*)$ is a normal subgroup of a group $(G,\*)$ for all $i=1,…,n$, then the chain $G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$ is called a *normal chain*.

**Example(10-6):**

Find all chains in the following groups and determine their length and type.

* $(Ζ\_{6}, +\_{6})$;
* $(Ζ\_{8}, +\_{8})$;
* $(Ζ\_{18}, +\_{18})$ (**Homework**);
* $(Ζ\_{21}, +\_{21})$ (**Homework).**

**Solution:** The subgroups of a group $(Ζ\_{6}, +\_{6})$ are :

$$H\_{1}=(Ζ\_{6}, +\_{6})$$

$$H\_{2}=(\{0\}, +\_{6})$$

$$H\_{3}=\left(\left〈2\right〉, +\_{6}\right)=(\left\{0,2,4\right\},+\_{6})$$

$$H\_{4}=\left(\left〈3\right〉, +\_{6}\right)=(\left\{0,3\right\},+\_{6})$$

Then the chains in $(Ζ\_{6}, +\_{6})$ are:

$Ζ\_{6}⊃\{0\}$ is a trivial chain of length one

$Ζ\_{6}⊃\left〈2\right〉⊃\{0\}$ is a normal chain of length two

$Ζ\_{6}⊃\left〈3\right〉⊃\{0\}$ is a normal chain of length two.

The subgroups of a group $(Ζ\_{8}, +\_{8})$ are :

$$H\_{1}=(Ζ\_{8}, +\_{8})$$

$$H\_{2}=(\{0\}, +\_{8})$$

$$H\_{3}=\left(\left〈2\right〉, +\_{8}\right)=(\left\{0,2,4,6\right\},+\_{8})$$

$$H\_{4}=\left(\left〈4\right〉, +\_{6}\right)=(\left\{0,4\right\},+\_{8})$$

Then the chains in $(Ζ\_{8}, +\_{8})$ are:

$Ζ\_{8}⊃\{0\}$ is a trivial chain of length one

$Ζ\_{8}⊃\left〈2\right〉⊃\{0\}$ is a normal chain of length two

$Ζ\_{8}⊃\left〈4\right〉⊃\{0\}$ is a normal chain of length two

$Ζ\_{8}⊃\left〈2\right〉⊃\left〈4\right〉⊃\{0\}$ is a normal chain of length three.

 **Definition(10-7):** (***Composition Chain***)

In the group $(G,\*)$, the descending sequence of sets

$$G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$$

 forms a *composition chain* for $(G,\*)$ provided

1. $(H\_{i},\*)$ is a subgroup of $(G,\*)$,
2. $(H\_{i},\*)$ is a normal subgroup of $(H\_{i-1},\*)$,
3. The inclusion $H\_{i-1}⊇K⊇H\_{i}$, where $(K,\*)$ is a normal subgroup of $(H\_{i-1},\*)$, implies either $ K=H\_{i-1}$ or $K=H\_{i}$.

**Remark(10-8):**

Every composition chain is a normal, but the converse is not true in general, the following example shows that.

**Example(10-9):**

In the group $(Ζ\_{24}, +\_{24})$, the normal chain

$$Ζ\_{24}⊃\left〈2\right〉⊃\left〈12\right〉⊃\{0\}$$

is not a composition chain, since it may be further refined by inserting of the set$\left〈4\right〉$ or $\left〈6\right〉$. On other hand,

$$Ζ\_{24}⊃\left〈2\right〉⊃\left〈4\right〉⊃\left〈8\right〉⊃\{0\}$$

and

$$Ζ\_{24}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$$

are both composition chains for $(Ζ\_{24}, +\_{24})$.

**Example(10-10):**

Find all chains in the following groups and determine their length and type.

* $(Ζ\_{8}, +\_{8})$;
* $(Ζ\_{12}, +\_{12})$;
* $(Ζ\_{18}, +\_{18})$ (**Homework**).

**Solution:** The subgroups of a group $(Ζ\_{8}, +\_{8})$ are :

$$H\_{1}=(Ζ\_{8}, +\_{8})$$

$$H\_{2}=(\{0\}, +\_{8})$$

$$H\_{3}=\left(\left〈2\right〉, +\_{8}\right)=(\left\{0,2,4,6\right\},+\_{8})$$

$$H\_{4}=\left(\left〈4\right〉, +\_{8}\right)=(\left\{0,4\right\},+\_{8})$$

Then the chains in $(Ζ\_{8}, +\_{8})$ are:

$Ζ\_{8}⊃\{0\}$ is a trivial chain of length one.

$Ζ\_{8}⊃\left〈2\right〉⊃\{0\}$ is a normal chain of length two, but it is not composition chain, since there is a normal subgroup $\left〈4\right〉$ in $Ζ\_{8}$, such that $\left〈2\right〉⊃\left〈4\right〉$.

$Ζ\_{8}⊃\left〈4\right〉⊃\{0\}$ is a normal chain of length two, but it is not composition chain, since there is a normal subgroup $\left〈2\right〉$ in $Ζ\_{8}$, such that $\left〈2\right〉⊃\left〈4\right〉$.

$Ζ\_{8}⊃\left〈2\right〉⊃\left〈4\right〉⊃\{0\}$ is a composition chain of length three.

The subgroups of a group $(Ζ\_{12}, +\_{12})$ are :

$$H\_{1}=(Ζ\_{12}, +\_{12})$$

$$H\_{2}=(\{0\}, +\_{12})$$

$$H\_{3}=\left(\left〈2\right〉, +\_{12}\right)=(\left\{0,2,4,6,8,10\right\},+\_{12})$$

$$H\_{4}=\left(\left〈3\right〉, +\_{12}\right)=(\left\{0,3,6,9\right\},+\_{12})$$

$$H\_{5}=\left(\left〈4\right〉, +\_{12}\right)=(\left\{0,4,8\right\},+\_{12})$$

$$H\_{6}=\left(\left〈6\right〉, +\_{12}\right)=(\left\{0,6\right\},+\_{12})$$

Then the chains in $(Ζ\_{12}, +\_{12})$ are:

$Ζ\_{12}⊃\{0\}$ is a trivial chain of length one.

$Ζ\_{12}⊃\left〈2\right〉⊃\{0\}$ is a normal chain of length two.

$Ζ\_{12}⊃\left〈3\right〉⊃\{0\}$ is a normal chain of length two.

$Ζ\_{12}⊃\left〈4\right〉⊃\{0\}$ is a normal chain of length two.

$Ζ\_{12}⊃\left〈6\right〉⊃\{0\}$ is a normal chain of length two.

$Ζ\_{12}⊃\left〈2\right〉⊃\left〈4\right〉⊃\{0\}$ is a composition chain of length three.

$Ζ\_{12}⊃\left〈3\right〉⊃\left〈6\right〉⊃\{0\}$ is a composition chain of length three.

**Example(10-11):**

Let $(G,\*)$ be the group of symmetries of the square.

A normal chain for $(G,\*)$ which fails to be a composition chain is

$G⊃\{R\_{180}, R\_{360}\}⊃\{R\_{360}\}$.

**Example(10-12):** (**Homework)**

Determine the following chain whether normal, composition:

$G⊃\{R\_{90}, R\_{180},R\_{270},R\_{360}\}⊃\{R\_{180},R\_{360}\}⊃\{R\_{360}\}$.

**Example(10-13):**

The group $(Ζ,+)$ has no a composition chain, since the normal subgroups of $(Ζ,+)$ are the cyclic subgroups $(\left〈n\right〉),+)$, $n$ a nonnegative integer, Since the inclusion $\left〈kn\right〉⊆\left〈n\right〉$ holds for all $k\in Ζ\_{+}$, there always exists a proper subgroup of any given group.

**Definition(10-14):**

A normal subgroup $(H,\*)$ is called a *maximal normal subgroup* of the group $(G,\*)$ if $H\ne G$ and there exists no normal subgroup $(K,\*)$ of $(G,\*)$ such that $H⊂K⊂G$.

**Example(10-15):**

In the group $(Ζ\_{24}, +\_{24})$, the cyclic subgroups $(\left〈2\right〉, +\_{24})$ and $(\left〈3\right〉, +\_{24})$ are both maximal normal with orders $12$ and $8$, respectively.

**Example(10-16):**

Determine the maximal normal subgroups in the group $(Ζ\_{12}, +\_{12})$.

**Solution:** The normal subgroups of $(Ζ\_{12}, +\_{12})$ are:

$$H\_{1}=\left(\left〈2\right〉, +\_{12}\right)=(\left\{0,2,4,6,8,10\right\},+\_{12})$$

$$H\_{2}=\left(\left〈3\right〉, +\_{12}\right)=(\left\{0,3,6,9\right\},+\_{12})$$

$$H\_{3}=\left(\left〈4\right〉, +\_{12}\right)=(\left\{0,4,8\right\},+\_{12})$$

$$H\_{4}=\left(\left〈6\right〉, +\_{12}\right)=(\left\{0,6\right\},+\_{12})$$

The maximal normal subgroups of $(Ζ\_{12}, +\_{12})$ are $H\_{1}$ and $H\_{2}$, since there is no normal subgroup in $Ζ\_{12}$ containing $H\_{1}$ and $H\_{2}$.

**Remark(10-17):**

A chain $G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$ is a composition of a group $(G,\*)$, if each normal subgroup $(H\_{i},\*)$ is a maximal normal subgroup of $(H\_{i-1},\*)$, for all $i=1,…,n$.

**Example(10-18);**

In the group $(Ζ\_{12}, +\_{12})$ the chains $Ζ\_{12}⊃\left〈2\right〉⊃\left〈4\right〉⊃\{0\}$ is a composition of$ Ζ\_{12}$ , since

 $\left〈2\right〉$ is a maximal normal subgroup of $Ζ\_{12}, $

 $\left〈4\right〉 $is a maximal normal subgroup of$ \left〈2\right〉$,

$\{0\}$ is a maximal normal subgroup of$ \left〈4\right〉$, and

$Ζ\_{12}⊃\left〈3\right〉⊃\left〈6\right〉⊃\{0\}$ is a composition of $Ζ\_{12}$, since

$\left〈3\right〉$ is a maximal normal subgroup of $Ζ\_{12}, $

 $\left〈6\right〉 $is a maximal normal subgroup of$ \left〈3\right〉$,

$\{0\}$ is a maximal normal subgroup of$ \left〈6\right〉$.

**Theorem(10-19):**

A normal subgroup $(H,\*)$ of the group $(G,\*)$ is a maximal if and only if the quotient $(^{G}/\_{H},⊗)$ is a simple.

**Proof:**

$⇒)$ Let $H⊵ K⟹\frac{K}{H}⊵\frac{G}{H}⟹H=K $or$ K=G$

Since $H$ is a maximal, $⟹\frac{K}{H}=H $or$ \frac{K}{H}=\frac{G}{H} ⟹\frac{G}{H}$ is a simple

$⇐)$ let $^{G}/\_{H}$ be a simple

 $⇒$ $^{G}/\_{H}$ has two normal subgroups which are $e\*H$ and $^{G}/\_{H}$, but $e\*H=H$

Therefore $H$ is a maximal$ ∎$

**Corollary(10-20):**

The group $(^{G}/\_{H},⊗)$ is a simple, if $\left|^{G}/\_{H}\right|$ is a prime number.

**Examples(10-21);**

1. Show that $(\left〈2\right〉, +\_{12})$ is a maximal normal subgroup of $(Ζ\_{12}, +\_{12})$.
2. Show that $(\left〈3\right〉, +\_{15})$ is a maximal normal subgroup of $(Ζ\_{15}, +\_{15})$. (**Homework**)

**Solution(1):** $\left(\left〈2\right〉, +\_{12}\right)=(\left\{0,2,4,6,8,10\right\},+\_{12})$

$\left|^{G}/\_{H}\right|=\frac{\left|G\right|}{\left|H\right|}=\frac{\left|Ζ\_{12}\right|}{\left|\left〈2\right〉\right|}=\frac{12}{6}=2$ is a prime $⇒\frac{Ζ\_{12}}{\left〈2\right〉}$ is a simple (by Corollary (10-20)). From Theorem (10-19), we get that $\left〈2\right〉$ is a maximal normal subgroup of $ Ζ\_{12}$.

**Corollary(10-22):**

A normal chain $G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$ is a composition of a group $(G,\*)$, if $(^{H\_{i}}/\_{H\_{i-1}},⊗)$ is a simple group for all$ i=1,…,n$.

**Example(10-23);**

Show that $Ζ\_{60}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$ is a composition chain of a group $(Ζ\_{60}, +\_{60})$.

**Solution:** $\frac{\left|Ζ\_{60}\right|}{\left|\left〈3\right〉\right|}=\frac{60}{20}=3$ is a prime $⇒\frac{Ζ\_{60}}{\left〈3\right〉}$ is a simple.

So, we get that $\left〈3\right〉 $is a maximal normal subgroup of $ Ζ\_{60}$.

$\frac{\left|\left〈3\right〉\right|}{\left|\left〈6\right〉\right|}=\frac{20}{10}=2$ is a prime $⇒\frac{\left〈3\right〉}{\left〈6\right〉}$ is a simple.

So, we get that $\left〈6\right〉 $is a maximal normal subgroup of $\left〈3\right〉$.

$\frac{\left|\left〈6\right〉\right|}{\left|\left〈12\right〉\right|}=\frac{10}{5}=2$ is a prime $⇒\frac{\left〈6\right〉}{\left〈12\right〉}$ is a simple.

So, we get that $\left〈12\right〉$ is a maximal normal subgroup of $\left〈6\right〉$.

$\frac{\left|\left〈12\right〉\right|}{\left|\{0\}\right|}=\frac{5}{1}=5$ is a prime $⇒\frac{\left〈12\right〉}{\{0\}}$ is a simple.

So, we get that $\{0\}$ is a maximal normal subgroup of $\left〈12\right〉$.

By corollaries (10-19) and (1-21), we have that $Ζ\_{60}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$ is a composition chain of a group $(Ζ\_{60}, +\_{60})$.

**Theorem(10-24):**

Every finite group $(G,\*)$ with more than one element has a composition chain.

**Theorem(10-25):** (**Jordan-Holder**)

In a finite group $(G,\*)$ with more than one element, any two composition chains are equivalent.

**Example(10-26):**

In a group $(Ζ\_{60}, +\_{60})$, show that the two chains

$$Ζ\_{60}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$$

$Ζ\_{60}⊃\left〈2\right〉⊃\left〈6\right〉⊃\left〈30\right〉⊃\{0\}$,

are compositions and equivalent.

**Solution:**

$(^{Ζ\_{60}}/\_{\left〈3\right〉}, ⊗)≅(^{\left〈2\right〉}/\_{\left〈6\right〉}, ⊗)$, since $\left|^{Ζ\_{60}}/\_{\left〈3\right〉}\right|=\frac{60}{20}=3=\left|^{\left〈2\right〉}/\_{\left〈6\right〉}\right|=\frac{30}{10}$,

$(^{\left〈3\right〉}/\_{\left〈6\right〉}, ⊗)≅(^{Ζ\_{60}}/\_{\left〈2\right〉}, ⊗)$, since $\left|^{\left〈3\right〉}/\_{\left〈6\right〉}\right|=\frac{20}{10}=2=\left|^{Ζ\_{60}}/\_{\left〈2\right〉}\right|=\frac{60}{30}$,

$(^{\left〈6\right〉}/\_{\left〈12\right〉}, ⊗)≅(^{\left〈30\right〉}/\_{\{0\}}, ⊗)$, since $\left|^{\left〈6\right〉}/\_{\left〈12\right〉}\right|=\frac{10}{5}=2=\left|^{\left〈30\right〉}/\_{\{0\}}\right|=\frac{2}{1}$,

$(^{\left〈12\right〉}/\_{\{0\}}, ⊗)≅(^{\left〈6\right〉}/\_{\left〈30\right〉}, ⊗)$, since $\left|^{\left〈12\right〉}/\_{\{0\}}\right|=\frac{5}{1}=5=\left|^{\left〈6\right〉}/\_{\left〈30\right〉}\right|=\frac{10}{2}$.

Therefore, by Jordan-Holder theorem the two chains

$$Ζ\_{60}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$$

$Ζ\_{60}⊃\left〈2\right〉⊃\left〈6\right〉⊃\left〈30\right〉⊃\{0\}$,

are compositions and equivalent.

**Exercises(10-27):**

* Check that the following chains represent composition chains for the indicated group.
1. For $(Ζ\_{36}, +\_{36})$, the group of integers modulo $36$:

$Ζ\_{36}⊃\left〈3\right〉⊃\left〈9\right〉⊃\left〈18\right〉⊃\{0\}$.

1. For $(G\_{s},\*)$, the group of symmetries of the square:

$G⊃\{R\_{180}, R\_{360},D\_{1},D\_{2}\}⊃\{R\_{360},D\_{1}\}⊃\{R\_{360}\}$.

1. For $(\left〈a\right〉,\*)$, a cyclic group of order$ 30$:

$\left〈a\right〉⊃\left〈a^{5}\right〉⊃\left〈a^{10}\right〉⊃\{e\}$.

1. For $(S\_{3},∘)$, the symmetric group on $3$ symbols:

$S\_{3}⊃\left\{i,\left(123\right),\left(132\right)\right\}⊃\{i\}$.

* Find a composition chain for the symmetric group $(S\_{4},∘)$.
* Prove that the cyclic subgroup $(\left〈n\right〉,+)$ is a maximal normal subgroup of $(Ζ,+)$ if and only if $n$ is a prime number.
* Establish that the following two composition chains for $(Ζ\_{36}, +\_{36})$ are equivalent:

$Ζ\_{24}⊃\left〈3\right〉⊃\left〈6\right〉⊃\left〈12\right〉⊃\{0\}$,

$Ζ\_{24}⊃\left〈2\right〉⊃\left〈4\right〉⊃\left〈12\right〉⊃\{0\}$.

* Find all composition chains for $(Ζ\_{36}, +\_{36})$.
* Find all composition chains for $(G\_{s},\*)$.
1. $ P$**- Groups and Related Concepts.**

**Definition(11-1):** ($p$**- Group**)

A finite group $(G,\*)$ is said to be $p$*- group* if and only if the order of each element of $G$ is a power of fixed prime $p$.

**Definition(11-2):** ($p$**- Group**)

 A finite group $(G,\*)$ is said to be $p$*- group* if and only if $\left|G\right|=p^{k},k\in Ζ$, where $p$ is a prime number.

**Example(11-3):**

Show that $(Ζ\_{4}, +\_{4})$ is a $p$- group.

**Solution:** $ Ζ\_{4}=\{0,1,2,3\}$ and$ \left| Ζ\_{4}\right|=4=2^{2}$

$⇒$ $ Ζ\_{4}$ is a $2$- group, with

$o(0)=1=2^{0}$,

$o(1)=4=2^{2}$,

$o(2)=2=2^{1}$,

$o(3)=4=2^{2}$.

**Example(11-4):**

Determine whether $(Ζ\_{6}, +\_{6})$ is a $p$- group.

**Solution:** $ Ζ\_{6}=\{0,1,2,3,4,5\}$ and$ \left| Ζ\_{6}\right|=6\ne P^{k}$

$⇒$ $ Ζ\_{6}$ is not $p$- group.

**Example(11-5):** (**Homework)**

Determine whether $(G\_{s}, ∘)$ is a $p$- group.

**Examples(11-6):**

* $(Ζ\_{8}, +\_{8})$ is a $2$- group, since $\left|Ζ\_{8}\right|=8=2^{3}$,
* $(Ζ\_{9}, +\_{9})$ is a $3$- group, since $\left|Ζ\_{9}\right|=9=3^{2}$,
* $(Ζ\_{25}, +\_{25})$ is a $5$- group, since $\left|Ζ\_{25}\right|=25=5^{2}$.

**Theorem(11-7):**

Let $H∆G$, then $G$ is a $p$- group if and only if $ H$ and $^{G}/\_{H}$ are $p$- groups.

**Proof:** $(⟹)$ Assume that $G$ is a $p$- group, to prove that $H$ and $^{G}/\_{H}$ are $p$- groups.

Since $G$ is a $p$- group$ ⟹$ $o\left(a\right)=p^{x}$, for some $x\in Ζ^{+}, ∀a\in G$.

Since $H⊆G$ $⟹∀a\in H$ group$ ⟹$ $o\left(a\right)=p^{x}$, for some $x\in Ζ^{+}$.

So, $H$ is a $p$- group.

To prove $^{G}/\_{H}$ is a $p$- group.

Let $ a\*H\in ^{G}/\_{H}$, to prove $o(a\*H)$ is a power of $p$.

$(a\*H)^{p^{x}}=a^{p^{x}}\*H=e\*H=H$, ($a^{p^{x}}=e$ since $G$ is a $p$- group

$⟹o\left(a\*H\right)=p^{x}$

$ (⟸)$ Suppose that $H$ and $^{G}/\_{H}$ are $p$- groups, to prove $G$ is a $p$- group.

Let $a\in G$, to prove $o(a)$ is a power of $p$.

$(a\*H)^{p^{x}}=H…(1)$ ($^{G}/\_{H}$ is a $p$- group)

$$(a\*H)^{p^{x}}=a^{p^{x}}\*H…(2)$$

From (1) and (2), we have $a^{p^{x}}\*H=H⟹a^{p^{x}}\in H$ and $H$ is a $p$- group,

$$⟹o\left(a^{p^{x}}\right)=p^{r}, r\in Ζ^{+}$$

$⟹\left(a^{p^{x}}\right)^{p^{r}}=e⟹a^{p^{x+r}}=e, x+r\in Ζ^{+}$,

$$⟹o\left(a\right)=p^{x+r}$$

Therefore, $G$ is a $p$- group$ ∎$

**Examples(11-8):**

Apply theorem(2-7) on $(Ζ\_{32}, +\_{32})$.

**Solution:**

$\left|Ζ\_{32}\right|=32=2^{5}$ is a $2$- group.

By theorem (2-7), $H$ and $^{G}/\_{H}$ are $2$- groups.

$^{o(G)}/\_{o(H)}$ $⟹o\left(H\right)=2^{x},0\leq x\leq 5$.

$o\left(H\right)=2^{0}$ or $2^{1}$ or $2^{2}$ or $2^{3}$ or $2^{4}$ or $2^{5}$,

$o\left(H\right)=2^{0}$ is a $2$- group $⟹o\left(^{G}/\_{H}\right)=^{o\left(G\right)}/\_{o\left(H\right)}=\frac{2^{5}}{2^{0}}=2^{5}$ is a $2$- group.

$o\left(H\right)=2^{1}$ is a $2$- group $⟹^{o\left(G\right)}/\_{o\left(H\right)}=2^{4}$

$o\left(H\right)=2^{2}$ is a $2$- group $⟹^{o\left(G\right)}/\_{o\left(H\right)}=2^{3}$

$o\left(H\right)=2^{3}$ is a $2$- group $⟹^{o\left(G\right)}/\_{o\left(H\right)}=2^{2}$

$o\left(H\right)=2^{4}$ is a $2$- group $⟹^{o\left(G\right)}/\_{o\left(H\right)}=2$

$o\left(H\right)=2^{5}$ is a $2$- group $⟹^{o\left(G\right)}/\_{o\left(H\right)}=1$.

**Remark(11-9);**

If $G$ is a non-trivial $p$- group, then Cent$(G)\ne e$.

**Theorem(11-10):**

Every group of order $p^{2}$ is an abelian.

**Proof:** Let $G$ be a group of order $p^{2}$, to prove $G$ is an abelian.

Let Cent$(G)$ is a subgroup of $G$.

By Lagrange Theorem $^{o\left(G\right)}/\_{o\left(Cent(G) \right)}$ ,

$$⟹^{p^{2}}/\_{o\left(Cent(G) \right)}$$

$⟹o\left(Cent\left(G\right)\right)=p^{0}$ or $p^{1}$ or $p^{2}$

If $o\left(Cent\left(G\right)\right)=p^{0}$ $⟹Cent\left(G\right)=\{e\}$, but this is contradiction with remark(2-9), so $o\left(Cent\left(G\right)\right)\ne p^{0}$.

If $o\left(Cent\left(G\right)\right)=p^{2}=o\left(G\right)⟹Cent\left(G\right)=G$

$⟹G$ is an abelian.

If $o\left(Cent\left(G\right)\right)=p^{1}⟹o\left(^{G}/\_{Cent\left(G\right)}\right)=\frac{p^{2}}{p^{1}}=p$

$^{G}/\_{Cent\left(G\right)}$ is a cyclic.

Therefore, $G$ is an abelian $∎$

**Remark(11-11):**

The converse of theorem(2-10) is not true in general, for example $(Ζ\_{8}, +\_{8})$ is an abelian, but $o\left((Ζ\_{8}\right)=2^{3}\ne p^{2}$.

**Exercises(11-12):**

* Let $P$ and $Q$ be two normal $p$-subgroups of a finite group $G$. Show that $PQ$ is a normal $p$-subgroup of $G$.
* Determine whether $(Ζ\_{125}, +\_{125})$ is a $p$-group.
* Determine whether $(Ζ\_{121}, +\_{121})$ is a $p$-group.
* Determine whether $(Ζ\_{41}, +\_{41})$ is a $p$-group.
* Determine whether $(Ζ\_{16}, +\_{16})$ is a $p$-group.
* Determine whether $(Ζ\_{625}, +\_{625})$ is a $p$-group.
* Determine whether $(Ζ\_{185}, +\_{185})$ is a $p$-group.
* Determine whether $(Ζ\_{128}, +\_{128})$ is a $p$-group.
* Determine whether $(Ζ\_{256}, +\_{256})$ is a $p$-group.
* Determine whether $(Ζ\_{100}, +\_{100})$ is a $p$-group.
* Show that $G\_{l}=\{\pm 1,\pm i,\pm j,\pm k\}, ∙)$ is a $p$-group.
1. **Sylow Theorems**

**Definition(12-1):** (***Sylow*** $p$***- Subgroup***)

Let $(G,\*)$ be a finite group and $p$ is a prime number, a subgroup $(H,\*)$ of a group $G$ is called *sylow* $p$*- subgroup* if

1. $(H,\*)$ is a $p$- group,
2. $(H,\*)$ is not contained in any other $p$- subgroup of $G$ for the same prime number $p$.

**Example(12-2);**

Find sylow $2$- subgroups and sylow$ 3$- subgroup of the group $(Ζ\_{24}, +\_{24})$.

**Solution:** The proper subgroups of the group $(Ζ\_{24}, +\_{24})$ are

1. $\left(\left〈2\right〉, +\_{24}\right)⟹o\left(\left〈2\right〉\right)=12\ne P^{k}⟹\left〈2\right〉$ is not $p$- subgroup.
2. $\left(\left〈3\right〉, +\_{24}\right)⟹o\left(\left〈3\right〉\right)=8=2^{3}⟹\left〈3\right〉$ is a $2$- subgroup.
3. $\left(\left〈4\right〉, +\_{24}\right)⟹o\left(\left〈4\right〉\right)=6\ne P^{k}⟹\left〈4\right〉$ is not $p$- subgroup.
4. $\left(\left〈6\right〉, +\_{24}\right)⟹o\left(\left〈6\right〉\right)=4=2^{2}⟹\left〈6\right〉$ is a $2$- subgroup.
5. $\left(\left〈8\right〉, +\_{24}\right)⟹o\left(\left〈8\right〉\right)=3=3^{1}⟹\left〈8\right〉$ is a $3$- subgroup.
6. $\left(\left〈12\right〉, +\_{24}\right)⟹o\left(\left〈12\right〉\right)=2=2^{1}⟹\left〈12\right〉$ is a $2$- subgroup.

**Theorem(12-3):** (**First Sylow Theorem**)

Let $(G,\*)$ be a finite group of order $p^{k}q$, where $p$ is a prime number is not dividing $q$, then $G$ has sylow $p$- subgroup of order $p^{k}$.

**Example(12-4):**

Find sylow $2$- subgroup of the group $(Ζ\_{12}, +\_{12})$.

**Solution:** $o(Ζ\_{12})=12=\left(4\right)\left(3\right)=(2^{2})(3)$, and $2∤3$

$⟹$ by first sylow theorem, the group $(Ζ\_{12}, +\_{12})$ has sylow $2$- subgroup of order $2^{2}$.

$⟹$ $(\left〈3\right〉, +\_{12})$ is a sylow $2$- subgroup.

**Example(12-5):**

Find sylow $7$- subgroup of the group $(Ζ\_{42}, +\_{42})$.

**Solution:** $o(Ζ\_{42})=42=\left(7\right)\left(6\right)$, and $7∤6$

$⟹$ by first sylow theorem, the group $(Ζ\_{42}, +\_{42})$ has sylow $7$- subgroup of order $7^{1}$.

$⟹$ $(\left〈6\right〉, +\_{42})$ is a sylow $7$- subgroup.

**Example(12-6):**

Find sylow $3$- subgroup of the group $(Ζ\_{24}, +\_{24})$.

**Solution:** $o(Ζ\_{24})=24=\left(3\right)\left(8\right)=(3^{1})(8)$, and $3∤8$

$⟹$ by first sylow theorem, the group $(Ζ\_{24}, +\_{24})$ has sylow $3$- subgroup of order $3^{1}$.

$⟹$ $(\left〈8\right〉, +\_{24})$ is a sylow $3$- Subgroup.

**Theorem(12-7):**

Let $p$ a prime number and $G$ be a finite group such that $p^{x}\o\left(G\right), x\geq 1$, then $G$ has a subgroup of order $p^{x}$ which is called sylow $p$- subgroup of $G$.

**Example(12-8):**

Are the following groups $\left(S\_{3},∘\right)$ and $\left(G\_{s},∘\right) $have sylow $p$- subgroups.

**Solution:**

$\left(S\_{3},∘\right)$, $O(S\_{3})=6=\left(2\right)\left(3\right)$,

$2∖6⟹∃$ a subgroup $H$ such that $o\left(H\right)=2$ which is called sylow $2$- subgroup.

Also, $3∖6⟹∃$ a subgroup $K$ such that $o\left(K\right)=3$ which is called sylow $3$- subgroup.

$\left(G\_{s},∘\right)$, $o(G\_{s})=2^{3}$ is $2$- subgroup.

Every subgroup of $G\_{s} $is $2$- subgroup, $o\left(H\right)=2^{0}$ or $2^{1}$ or $2^{2}$ or $2^{3}$ .

**Theorem(12-9):** (**Second Sylow Theorem)**

The number of distinct sylow $p$-subgroups is $ k=1+tp,t=0,1,…$ which is divide the order of $G$.

**Example(12-10):**

Find the distinct sylow $p$-subgroups of $\left(S\_{3},∘\right)$.

**Solution:**

$o(S\_{3})=6=\left(2\right)\left(3\right)$,

$2∖6⟹∃$ a subgroup $H$ such that $o\left(H\right)=2$.

The number of sylow $2$-subgroups is $k\_{1}=1+2t,t=0,1,…$ and $k\_{1}∖6$

if $t=0⟹k\_{1}=1$ and $1∖6$

if $t=1⟹k\_{1}=3$ and $3∖6$

if $t=2⟹k\_{1}=5$ and $5∤6$

if $t=3⟹k\_{1}=7$ and $7∤6$

so, there are two sylow $2$-subgroups.

$3∖6⟹∃$ a subgroup $K$ such that $o\left(K\right)=3$.

The number of sylow $3$-subgroups is $k\_{2}=1+3t,t=0,1,…$ and $k\_{2}∖6$

if $t=0⟹k\_{2}=1$ and $1∖6$

if $t=1⟹k\_{2}=4$ and $4∤6$

if $t=2⟹k\_{2}=7$ and $7∤6$

So, there is one sylow $3$-subgroup.

**Example(12-11):**

Find the number of sylow $p$-subgroups of $G$ such that $o\left(G\right)=12.$

**Solution:** $o\left(G\right)=12=(3)(2^{2})$

$3∖12⟹∃$ a subgroup $H$ such that $o\left(H\right)=3$.

The number of sylow $3$-subgroups is $k\_{1}=1+3t,t=0,1,…$ and $k\_{1}∖12$

if $t=0⟹k\_{1}=1$ and $1∖12$

if $t=1⟹k\_{1}=4$ and $4∖12$

if $t=2⟹k\_{1}=7$ and $7∤12$

if $t=3⟹k\_{1}=10$ and $10∤12$

So, there are two sylow $3$-subgroups of $G$.

The number of sylow $2$-subgroups is $k\_{2}=1+2t,t=0,1,…$ and $k\_{2}∖12$

if $t=0⟹k\_{2}=1$ and $1∖12$

if $t=1⟹k\_{2}=3$ and $3∖12$

if $t=2⟹k\_{2}=5$ and $5∤12$

if $t=3⟹k\_{2}=7$ and $7∤12$

So, there are two sylow $2$-subgroups of $G$.

**Remark(12-12):**

The group $G$ has exactly one sylow $p$-subgroup $H$ if and only if $H∆G$.

**Example(12-13):**

$$\left(S\_{3},∘\right), H=\{f\_{1}=i,f\_{2}=\left(123\right), f\_{3}=\left(132\right)\}$$

$H∆G⟹H$ is a sylow $3$-subgroup of $S\_{3}$,

So, there is one sylow $3$-subgroup of $S\_{3}$.

**Exercises(12-14);**

* Show that there is no simple group of order $200$.
* Show that there is no simple group of order $56$.
* Show that there is no simple group of order $20$.
* Show that whether $(G\_{l},∙)$ is a sylow.
1. **Solvable Groups and Their Applications**

**Definition(13-1):**

A group $(G,\*)$ is called a solvable group if and only if, there is a finite collection of subgroups of $(G,\*)$, $H\_{0}, H\_{1},…,H\_{n} $ such that

1. $G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$,
2. $H\_{i+1}∆H\_{i} ∀i=0,…,n-1$,
3. $^{H\_{i}}/\_{H\_{i+1}}$ is a commutative group $∀i=0,…,n-1$.

**Theorem(13-2):**

Every commutative group is a solvable group.

**Proof:**

Suppose that $(G,\*)$ is a commutative, to show that $(G,\*)$ is a solvable.

Let $G=H\_{0}$ and $H\_{1}=\{e\}$

1. $G=H\_{0}⊃H\_{1}=\{e\}$
2. $H\_{1}∆H\_{0}$ satisfies, since $\{e\}∆G$, or ( every subgroup of commutative group is a normal)
3. $^{G}/\_{\{e\}}≅G$ is a commutative group, or (the quotient of commutative group is a commutative)

So, $(G,\*)$ is a solvable group,

**Example(13-3):**

Show that $\left(S\_{3},∘\right)$ is a solvable group.

**Solution:** let $H\_{0}=S\_{3},H\_{1}=\left\{f\_{1}=i,f\_{2}=\left(123\right), f\_{3}=\left(132\right)\right\}, H\_{2}=\{f\_{1}\}$

1. $S\_{3}=H\_{0}⊃H\_{1}⊃H\_{2}=\{e\}$
2. $H\_{2}∆H\_{1}$ satisfies, since $\{ f\_{1}\}∆\{f\_{1},f\_{2},f\_{3}\}$, $H\_{1}∆H\_{0}$ is true,
3. To prove $^{H\_{i}}/\_{H\_{i+1}}$ is a commutative group $∀i=0,1$

 $o\left(^{H\_{1}}/\_{H\_{2}}\right)=\frac{o(H\_{1})}{o(H\_{2})}=\frac{3}{1}=3<6⟹^{H\_{1}}/\_{H\_{2}}$ is a commutative group

 $o\left(^{H\_{0}}/\_{H\_{1}}\right)=\frac{o(H\_{0})}{o(H\_{1})}=\frac{6}{3}=2<6⟹^{H\_{0}}/\_{H\_{1}}$ is a commutative group

Therefore, $\left(S\_{3},∘\right)$ is a solvable group.

**Example(13-4):** (**Homework)**

Show that $\left(G\_{s},∘\right)$ is a solvable group.

**Theorem(13-5):**

Every subgroup of a solvable group is a solvable.

**Proof:** let $(H,\*)$ be a subgroup of $(G,\*)$ and $(G,\*)$ is a solvable group.

To prove $(H,\*)$ is a solvable.

Since $G$ is a solvable $⟹$

there is a finite collection of subgroups of $(G,\*)$, $G\_{0}, G\_{1},…,G\_{n} $ such that

1. $G=G\_{0}⊃G\_{1}⊃…⊃G\_{n-1}⊃G\_{n}=\{e\}$,
2. $G\_{i+1}∆G\_{i} ∀i=0,…,n-1$,
3. $^{G\_{i}}/\_{G\_{i+1}}$ is a commutative group $∀i=0,…,n-1$.

Let $H\_{i}=H∩G\_{i}, i=0,…,n$

$$H\_{0}=H∩G\_{0}, H\_{1}=H∩G\_{1},…,H\_{n}=H∩G\_{n}=\{e\} $$

Each $H\_{i}$ is a subgroup of $(G,\*)$.

1. $H=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$ is hold
2. $H\_{i+1}∆H\_{i} ∀i=0,…,n-1$, $H\_{i}=H∩ G\_{i}, H\_{i+1}=H∩ G\_{i+1} $, since $G\_{i+1}∆G\_{i}⟹H\_{i+1}∆H\_{i}$
3. To prove $^{H\_{i}}/\_{H\_{i+1}}$ is a commutative group $∀i=0,…,n-1$.

Let $f\_{i}:H\_{i}⟶^{G\_{i}}/\_{G\_{i+1}},i=0,…,n-1 $such that $f\_{i}\left(x\right)=x\*G\_{i+1}∀x\in H\_{i}⊆G\_{i}$.

To prove $f\_{i}$ is a homomorphism,

$f\_{i}\left(x\*y\right)=f\_{i}(x)⊗f\_{i}(y)$ ?

$f\_{i}\left(x\*y\right)=x\*y\*G\_{i+1}=\left(x\*G\_{i+1}\right)⊗\left(y\*G\_{i+1}\right)=f\_{i}(x)⊗f\_{i}(y)$

So, $f\_{i}$ is a homomorphism

$f\_{i}$ is onto ?

$$R\_{f\_{i}}=\left\{f\_{i}\left(x\right):x\in H\_{i}\right\}=\left\{x\*G\_{i+1}:x\in H\_{i}\right\}=f\_{i}(H\_{i})\ne ^{G\_{i}}/\_{G\_{i+1}} $$

$f\_{i}(H\_{i})⊆^{G\_{i}}/\_{G\_{i+1}}⟹f\_{i}$ is not onto

$^{H\_{i}}/\_{kerf\_{i}⁡}≅f\_{i}(H\_{i})$ ( by theorem of homomorphism)

$$kerf\_{i}=\left\{x\in H\_{i}:f\_{i}\left(x\right)=e^{'}\right\}=\left\{x\in H\_{i}:x\*G\_{i+1}=G\_{i+1}\right\}=\left\{x\in H\_{i}:x\in G\_{i+1}\right\}=\left\{x\in H\_{i}:x\in H∩G\_{i+1}\right\}= H\_{i+1}$$

so, $\left(^{H\_{i}}/\_{H\_{i+1}},⊗\right)≅(f\_{i}\left(H\_{i}\right),⊗)$

 $f\_{i}\left(H\_{i}\right)⊆^{G\_{i}}/\_{G\_{i+1}}$ and $^{G\_{i}}/\_{G\_{i+1}}$ is a commutative

Hence,$ f\_{i}\left(H\_{i}\right)$ is a commutative

Therefore, $^{H\_{i}}/\_{H\_{i+1}}$ is a commutative

So, $(H,\*)$ is a solvable $∎$

**Theorem(13-6):**

Let $H∆G$ and $G$ is a solvable, then $^{G}/\_{H}$ is a solvable.

**Theorem(13-7):**

Let $H∆G$ and both $H, $ $^{G}/\_{H}$ are solvable, then $(G,\*)$ is a solvable.

**Proof:** since $(H,\*)$ is a solvable $⟹$

there is a finite collection of subgroups of $(G,\*)$, $H\_{0}, H\_{1},…,H\_{n} $ such that

1. $G=H\_{0}⊃H\_{1}⊃…⊃H\_{n-1}⊃H\_{n}=\{e\}$,
2. $H\_{i+1}∆H\_{i} ∀i=0,…,n-1$,
3. $^{H\_{i}}/\_{H\_{i+1}}$ is a commutative group $∀i=0,…,n-1$.

Since $(^{G}/\_{H}, ⊗)$ is a solvable $⟹$

there is a finite collection of subgroups of $(G,\*)$, $\frac{G\_{0}}{H}, \frac{G\_{1}}{H},…,\frac{G\_{r}}{H} $ such that

1. $\frac{G}{H}=\frac{G\_{0}}{H}⊃\frac{G\_{1}}{H}⊃…⊃\frac{G\_{r}}{H}=\left\{e\right\}=H$,
2. $\frac{G\_{i+1}}{H}∆\frac{G\_{i}}{H} ∀i=0,…,r-1$,
3. $^{\frac{G\_{i}}{H}}/\_{\frac{G\_{i+1}}{H}}$ is a commutative group $∀i=0,…,r-1$.

To prove $(G,\*)$ is a solvable group.

$$\frac{G}{H}=\frac{G\_{0}}{H}⟹G=G\_{0}$$

$\frac{G\_{r}}{H}=H⟹G\_{r}=\{e\}$ or $G\_{r}=H$

$$H∆G\_{r}⟹H⊆G\_{r}⟹G\_{r}=H$$

So, there is a finite collection $G\_{0}, G\_{1},…,G\_{r}=H\_{0}, H\_{1},…,H\_{n} $ such that

1. $G=G\_{0}⊃G\_{1}⊃…⊃G\_{r}=H=H\_{0}⊃H\_{1}⊃…⊃H\_{n}=\{e\}$.
2. To prove $G\_{i+1}∆G\_{i} ∀i=0,…,r-1$

Let $x\in G\_{i} $and $a\in G\_{i+1} $to prove $ x\*a\*x^{-1}\in G\_{i+1}$

$$x\in G\_{i}⟹x\*H\in \frac{G\_{i}}{H}$$

$$a\in G\_{i+1}⟹a\*H\in \frac{G\_{i+1}}{H}$$

$$\frac{G\_{i+1}}{H}∆\frac{G\_{i}}{H}⟹(x\*H)⊗(a\*H)⊗(x\*H)^{-1}\in \frac{G\_{i+1}}{H}$$

$$⟹(x\*a\*x^{-1})\*H\in \frac{G\_{i+1}}{H}⟹x\*a\*x^{-1}\in G\_{i+1}⟹G\_{i+1}∆G\_{i}$$

1. To prove $\frac{G\_{i}}{G\_{i+1}}$ is a commutative group $∀i=0,…,r-1$

$\frac{\frac{G\_{i}}{H}}{\frac{G\_{i+1}}{H}}$ is a commutative group and $\frac{\frac{G\_{i}}{H}}{\frac{G\_{i+1}}{H}}≅\frac{G\_{i}}{G\_{i+1}}$ ($\frac{\frac{G}{H}}{\frac{K}{H}}≅\frac{G}{K})$

$⟹\frac{G\_{i}}{G\_{i+1}}$ is a commutative group

Therefore, $(G,\*)$ is a solvable group $∎$

**Exercises(13-8);**

* Show that every $p$-group is a solvable group.
* Show that $\left(S\_{4},∘\right)$ is a solvable group.
* Show that $(Ζ\_{4}, +\_{4})$ is a solvable group.
* Show that $(Ζ\_{8}, +\_{8})$ is a solvable group.
* Show that $(Ζ\_{5}, +\_{5})$ is a solvable group.
* Show that $(Ζ\_{6}, +\_{6})$ is a solvable group.
* Show that $(Ζ\_{12}, +\_{12})$ is a solvable group.
* Show that $(Ζ\_{24}, +\_{24})$ is a solvable group.
1. **Applications of Group Theory**

**14-1 Cayley Theorem**

**Theorem(14-1-1): (Cayley Theorem)**

Every group is an isomorphic to a group of permutations.

This means if $(G,\*)$ is any group, then $(G,\*)≅(F\_{G},∘)$, where $F\_{G}=\left\{f\_{a}:a\in G\right\},f\_{a}:G⟶G \ni f\_{a}\left(x\right)=a\*x, ∀x\in G$.

**Proof:** define $g:G⟶F\_{G}$ by $g\left(a\right)=f\_{a}, ∀a\in G$

To prove $g$ is a homomorphism, one to one and onto.

1. $g$ is a homomorphism, let $a,b\in G$

$g\left(a\*b\right)=f\_{a\*b}=f\_{a}∘f\_{b}=g(a)∘g(b)⟹g$ is a homomorphism.

1. $g$ is a one to one, let$g\left(a\right)=g\left(b\right), ∀a,b\in G$

$$⟹f\_{a}=f\_{b}⟹f\_{a}\left(x\right)=f\_{b}\left(x\right)⟹a\*x=b\*x⟹a=b$$

$⟹g$ is a one to one.

1. $g$ is a onto,$ g\left(G\right)=\left\{g\left(a\right):a\in G\right\}=\left\{f\_{a}:a\in G\right\}=F\_{G}$

Therefore, $G≅F\_{G}∎$

**Corollary(14-1-2):**

Every finite group $(G,\*)$ of order $n$ is an isomorphic to $\left(S\_{n},∘\right)$.

**Example(14-1-3):**

Consider the following Cayley table of a group$ (G=\left\{e,a,b,c\right\},\*)$

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$\*$$ | $$e$$ | $$a$$ | $$b$$ | $$c$$ |
| $$e$$ | $$e$$ | $$a$$ | $$b$$ | $$c$$ |
| $$a$$ | $$a$$ | $$e$$ | $$c$$ | $$b$$ |
| $$b$$ | $$b$$ | $$c$$ | $$e$$ | $$a$$ |
| $$c$$ | $$c$$ | $$b$$ | $$a$$ | $$e$$ |

Show that$(G,\*)$ is an isomorphic to a subgroup of $\left(S\_{4},∘\right)$.

**Solution:**

$f\_{e}=\left(\begin{matrix}e&a\\e&a\end{matrix} \begin{matrix}b&c\\b&c\end{matrix}\right)$, $f\_{1}=\left(\begin{matrix}1&2\\1&2\end{matrix} \begin{matrix}3&4\\3&4\end{matrix}\right)=\left(1\right)\left(2\right)\left(3\right)\left(4\right)=(1)$

$f\_{a}=\left(\begin{matrix}e&a\\a&e\end{matrix} \begin{matrix}b&c\\c&b\end{matrix}\right)$, $f\_{2}=\left(\begin{matrix}1&2\\2&1\end{matrix} \begin{matrix}3&4\\4&3\end{matrix}\right)=\left(12\right)\left(34\right)$

$f\_{b}=\left(\begin{matrix}e&a\\b&c\end{matrix} \begin{matrix}b&c\\e&a\end{matrix}\right)$, $f\_{3}=\left(\begin{matrix}1&2\\3&4\end{matrix} \begin{matrix}3&4\\1&2\end{matrix}\right)=\left(13\right)\left(24\right)$

$f\_{c}=\left(\begin{matrix}e&a\\c&b\end{matrix} \begin{matrix}b&c\\a&e\end{matrix}\right)$, $f\_{4}=\left(\begin{matrix}1&2\\4&3\end{matrix} \begin{matrix}3&4\\2&1\end{matrix}\right)=\left(14\right)\left(23\right)$

Hence, $(G,\*)$ is an isomorphic to the subgroup of $\left(S\_{4},∘\right)$:

$\{\left(1\right),\left(12\right)\left(34\right),\left(13\right)\left(24\right),\left(14\right)\left(23\right)\}$.

**Example(14-1-4): (Homework)**

Let $ (G=\left\{1,-1,i,-i\right\},∙)$ be a group, apply Cayley Theorem on $G$.

**Example(14-1-5): (Homework)**

Show that $(Ζ\_{3}, +\_{3})$ is an isomorphic to a subgroup of $\left(S\_{3},∘\right)$.

**Exercises(14-1-6):**

* Apply Cayley Theorem on $(Ζ\_{4}, +\_{4})$.
* Apply Cayley Theorem on $(G=\left\{\pm 1,\pm i,\pm j,\pm k\right\},∙)$.
* Apply Cayley Theorem on$(G=\left\{1,-1\right\},∙)$.
* Apply Cayley Theorem on$(G=\{A=\left(\begin{matrix}1&0\\0&1\end{matrix}\right),B=\left(\begin{matrix}1&0\\0&-1\end{matrix}\right),C=\left(\begin{matrix}-1&0\\0&-1\end{matrix}\right),D=,\left(\begin{matrix}-1&0\\0&1\end{matrix}\right),∙)$.

**14-2 Direct Product**

**Definition(14-2-1):**

Let $(H,\*)$ and $(K,\*)$ be two normal subgroups of $(G,\*)$, then $(G,\*)$ is called an internal direct product of $H$ and $K$ ($G$ is a decomposition by $H$ and $K $) if and only if $G=H\*K$ and $H∩K=\{e\}$.

**Example(14-2-2):**

Consider the following Cayley table of a group$ \left(G=\left\{e,a,b,c\right\},\*\right), a^{2}=b^{2}=c^{2}=e$

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| $$\*$$ | $$e$$ | $$a$$ | $$b$$ | $$c$$ |
| $$e$$ | $$e$$ | $$a$$ | $$b$$ | $$c$$ |
| $$a$$ | $$a$$ | $$e$$ | $$c$$ | $$b$$ |
| $$b$$ | $$b$$ | $$c$$ | $$e$$ | $$a$$ |
| $$c$$ | $$c$$ | $$b$$ | $$a$$ | $$e$$ |

*Let* $H=\{e,a\}$ and $K=\{e,b\}$, show that $G=H⊗K$ is a decomposition by $H$ and $K$.

**Solution:** $H,K∆G$ since $G$ is a commutative group

$H\*K=\{e,a,b,c\}$ and $H∩K=\{e\}$

Hence,$ G=H⊗K$ is decomposition by $H$ and $K$.

**Example(14-2-3):**

*Let* $(G,\*)$be any group with$H=G$and$K=\{e\}$, show that

$G=H⊗K$ is a decomposition by $H$ and $K$.

**Solution:** $H,K∆G$

$$H\*K=G\*\left\{e\right\}=G$$

$$H∩K=G∩\left\{e\right\}=\{e\}$$

Therefore, $G=H⊗K$ is a decomposition by $H$ and $K$.

**Example(14-2-4):**

Let $(Ζ\_{4}, +\_{4})$ be a group. Is $ Ζ\_{4}$ has a proper decomposition.

**Solution:** the subgroups of $ Ζ\_{4}$ are $ Ζ\_{4}, \left\{0,2\right\}, \{0\}$

Let $ H=Ζ\_{4}$ and $K=\{0,2\}$

$$H⊗\_{4}K= Ζ\_{4}⊗\_{4}\left\{0,2\right\}= Ζ\_{4}$$

$$H∩K= Ζ\_{4}∩\left\{0,2\right\}=\{0,2\}$$

So, $ Ζ\_{4}\ne Ζ\_{4}⊗\{0,2\}$

Let $H=\{0\}$ and $K=\{0,2\}$

$$H⊗\_{4}K=K\ne Ζ\_{4}$$

Therefore, $ Ζ\_{4}$ has no proper decomposition.

**Theorem(14-2-5):**

Let $H$ and $K$ be two subgroups of $ G$ and $G=H⊗K$, then $^{G}/\_{H}≅K$ and $^{G}/\_{K}≅H$.

**Proof:**

Since $G=H⊗K⟹H\*K=G$ and $H∩K=\{e\}$

$^{G}/\_{H}=^{H\*K}/\_{H}$ and $^{H\*K}/\_{H}≅^{K}/\_{H∩K}$ (by second theorem of isomorphic)

$^{G}/\_{H}≅^{K}/\_{\{e\}}⟹^{G}/\_{H}≅K$ and

$^{G}/\_{K}=^{H\*K}/\_{K}$ and $^{H\*K}/\_{K}≅^{H}/\_{H∩K}$

$$^{G}/\_{K}≅^{H}/\_{\{e\}}⟹^{G}/\_{K}≅H∎$$

**Definition(14-2-6):**

Let $(G\_{1},\*)$ and $(G\_{2},∘)$ be two groups, define $G\_{1}×G\_{2}=\{\left(a,b\right):a\in G\_{1}, b\in G\_{2}\}$ such that $\left(a,b\right)⨀\left(c,d\right)=\left(a\*c,b∘d\right)\ni a,c\in G\_{1}, b,d\in G\_{2}$. Then $(G\_{1}×G\_{2},⨀)$ is a group which is called an external direct product of $G\_{1}$ and $G\_{2}$.

**Example(14-2-7): (Homework)**

Show that $(G\_{1}×G\_{2},⨀)$ is a group.

**Example(14-2-8):**

Let $G\_{1}=(Ζ\_{3}, +\_{3})$ and $G\_{2}=(Ζ\_{2}, +\_{2})$. Find $G\_{1}×G\_{2}$.

**Solution:**

$$G\_{1}×G\_{2}=Ζ\_{3}×Ζ\_{2}=\{\left(0,0\right),\left(0,1\right),\left(1,0\right),\left(1,1\right),\left(2,0\right),\left(2,1\right)\}$$

$$\left(1,1\right)⨀\left(2,1\right)=(0,0)$$

o$(Ζ\_{3}×Ζ\_{2})=o(Ζ\_{3}).o\left(Ζ\_{2}\right)=6$.

**Theorem(14-2-9):**

Let $(G\_{1},\*)$ and $(G\_{2},∘)$ be two groups, then

1. $(G\_{1}×G\_{2},⨀)$ is an abelian if and only if both $G\_{1}$ and $G\_{2}$ are abelian.
2. $G\_{1}×\{e\_{2}\}△G\_{1}×G\_{2}$.
3. $\{e\_{1}\}×G\_{2}△G\_{1}×G\_{2}$.
4. $G\_{1}≅G\_{1}×\{e\_{1}\}$.
5. $G\_{2}≅\{e\_{2}\}×G\_{2}$.

**Proof:**

1. $(⟹)$ suppose that $G\_{1}×G\_{2}$ is an abelian, to prove $G\_{1}$and $G\_{2}$ are abelian.

Let $\left(a,e\_{2}\right),\left(b,e\_{2}\right)\in G\_{1}×G\_{2}\ni a,b\in G\_{1}, e\_{2}\in G\_{2}$

Since $G\_{1}×G\_{2}$ is an abelian, then

$$\left(a,e\_{2}\right)⨀\left(b,e\_{2}\right)=\left(b,e\_{2}\right)⨀\left(a,e\_{2}\right)$$

$$\left(a\*b,e\_{2}\right)=\left(b\*a,e\_{2}\right)⟹a\*b=b\*a$$

Hence,$ (G\_{1},\*)$ is an abelian.

Similarly that $(G\_{2},\*)$ is an abelian.

$(⟸)$ suppose that $(G\_{1},\*)$ and $(G\_{2},∘)$ are abelian, to prove $G\_{1}×G\_{2}$ is an abelian.

Let $\left(a,b\right),\left(c,d\right)\in G\_{1}×G\_{2}$, to prove $\left(a,b\right)⨀\left(c,d\right)=(c,d)⨀(a,b)$

$$\left(a,b\right)⨀\left(c,d\right)=(a\*c,b∘d)$$

$$\left(c,d\right)⨀\left(a,b\right)=(c\*a,d∘b)$$

$a\*c= c\*a$ ($G\_{1}$is an abelian)

$b∘d= d∘b$ ($G\_{2}$is an abelian)

$$⟹\left(a,b\right)⨀\left(c,d\right)=(c,d)⨀(a,b)$$

Therefore, $G\_{1}×G\_{2}$ is an abelian.

1. To prove $G\_{1}×\{e\_{2}\}△G\_{1}×G\_{2}$

$$G\_{1}×\left\{e\_{2}\right\}=\{\left(a,e\_{2}\right):a\in G\_{1}\}\ne ∅$$

To prove $(G\_{1}×\left\{e\_{2}\right\},⨀)$ is a subgroup of $G\_{1}×G\_{2}$

Let $\left(a,e\_{2}\right), \left(b,e\_{2}\right)\in G\_{1}×\left\{e\_{2}\right\}$

$$\left(a,e\_{2}\right)⨀\left(b,e\_{2}\right)^{-1}=\left(a,e\_{2}\right)⨀\left(b^{-1},e\_{2}^{-1}\right)=(a\*b^{-1},e\_{2})$$

So, $(G\_{1}×\left\{e\_{2}\right\},⨀)$ is a subgroup of $G\_{1}×G\_{2}$.

To prove $G\_{1}×\{e\_{2}\}△G\_{1}×G\_{2}$

Let $(x,y)\in G\_{1}×G\_{2}$ and $(a,e\_{2})\in G\_{1}×\{e\_{2}\}$

To prove $(x,y)⨀(a,e\_{2})⨀(x,y)^{-1}\in G\_{1}×\{e\_{2}\}$

$$\left(x\*a\*x^{-1},y\*e\_{2}\*y^{-1}\right)=(x\*a\*x^{-1},e\_{2})\in G\_{1}×\{e\_{2}\}$$

Hence, $G\_{1}×\{e\_{2}\}△G\_{1}×G\_{2}$.

1. (**Homework).**
2. To prove $G\_{1}≅G\_{1}×\{e\_{2}\}$.

**Proof:**

Define $f: \left(G\_{1},\*\right)⟶\left(G\_{1}×\left\{e\_{2}\right\},⨀\right) \ni f\left(a\right)=(a,e\_{2})$

$f$ is a map ? let $ a\_{1}, a\_{2}\in G\_{1}$ and $a\_{1}=a\_{2}⟹\left(a\_{1},e\_{2}\right)=\left(a\_{2},e\_{2}\right)⟹f\left(a\_{1}\right)=f(a\_{2})$, so $f$ is a map

$f$ is an one to one ? let $ f\left(a\_{1}\right)=f(a\_{2})⟹\left(a\_{1},e\_{2}\right)=\left(a\_{2},e\_{2}\right)⟹a\_{1}=a\_{2}$, so $f$ is a one to one.

$f$ is a homomorphism ? $f\left(a\*b\right)=\left(a\*b, e\_{2}\right)=\left(a,e\_{2}\right)⨀\left(b,e\_{2}\right)=f(a)⨀f(b)$, so $f$ is a homomorphism

$f$ is an onto ? $R\_{f}=\left\{f\left(a\right):a\in G\_{1}\right\}=\left\{\left(a,e\_{2}\right):a\in G\_{1}\right\}=G\_{1}×\left\{e\_{2}\right\}$ so $f$ is an onto.

Therefore, $(G\_{1},\*)≅(G\_{1}×\left\{e\_{2}\right\},⨀)∎$

1. (**Homework)**

**Theorem(14-2-10):**

Let $(G\_{1},\*)$ and $(G\_{2},∘)$ be two $p$-groups, then$ (G\_{1}×G\_{2},⨀)$ is a $p$-group.

**Proof:**

Since $G\_{1}$is $p$-group $⟹o\left(G\_{1}\right)=p^{k\_{1}},k\_{1}\in Ζ^{+}$

Since $G\_{2}$is $p$-group $⟹o\left(G\_{2}\right)=p^{k\_{2}},k\_{2}\in Ζ^{+}$

$$o\left(G\_{1}×G\_{2}\right)=o\left(G\_{2}\right)×o\left(G\_{1}\right)=p^{k\_{1}}×p^{k\_{2}}=p^{k\_{1}+k\_{2}},k\_{1}+k\_{2}\in Ζ^{+}$$

Therefore, $ G\_{1}×G\_{2}$ is a $p$-group $∎$

**Exercises(14-2-11):**

* Let $H=\{0,2,4\}$and$ K=\{0,3\}$are subgroups of $(Ζ\_{6}, +\_{6})$, show that $Ζ\_{6}=H⊗K$ is a decomposition.
* Let $H=\{0\}$, show that $Ζ\_{7}=H⊗Ζ\_{7}$ is a decomposition.
* Find $Ζ\_{3}×Ζ\_{7}$.
* Is $S\_{3}×Ζ\_{2}$ an abelian?
* Is $G\_{s}×Ζ\_{2}$ an abelian?
* Is $S\_{3}×G\_{S}$ an abelian?
* Is $\{\pm 1,\pm i\}×Ζ\_{2}$ an abelian?
* Is $Ζ\_{4}×Ζ\_{8}$ a $p$-group?
* Is $Ζ\_{5}×Ζ\_{25}$ a $p$-group?
* Is $Ζ\_{11}×Ζ\_{121}$ a $p$-group?
* Is $Ζ\_{7}×Ζ\_{49}$ a $p$-group?
* Is $Ζ\_{27}×Ζ\_{3}$ a $p$-group?
* Is $Ζ\_{5}×Ζ\_{125}$ a $p$-group?
* Is $Ζ\_{2}×Ζ\_{64}$ a $p$-group?
* Is $Ζ\_{4}×Ζ\_{128}$ a $p$-group?
* Is $Ζ\_{9}×Ζ\_{81}$ a $p$-group?
* Is $Ζ\_{27}×Ζ\_{81}$ a $p$-group?
* Is $Ζ\_{128}×Ζ\_{8}$ a $p$-group?
* Is $Ζ\_{2}×Ζ\_{256}$ a $p$-group?