1. Chapter I: Matrices and Determinants

This chapter is an introduction to matrices and matrix operations. Determinants, Cramer's rule, and Gauss's elimination method are introduced. Some definitions and examples are not applicable to subsequent material presented in this text, but are included for subject continuity, and reference to more advanced topics in matrix theory.

1.1. Matrix Definition

A matrix is a rectangular array of numbers such as those shown below.

$$\begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 3 & 1 \\ -2 & 1 & -5 \\ 4 & -7 & 6 \end{bmatrix}$$

In general form, a matrix A is denoted as

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \quad (3.1)$$

The numbers a_{ij} are the elements of the matrix where the index i indicates the row and j indicates the column in which each element is positioned. Thus, a_{43} indicates the element positioned in the fourth row and third column.

A matrix of m rows and n columns is said to be of $m \times n$ *order matrix*.

If $m = n$, the matrix is said to be a square matrix of order m (or n). Thus, if a matrix has five rows and five columns, it is said to be a square matrix of order 5.
In a square matrix, the elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are called the *main diagonal elements*. Alternately, we say that the matrix elements $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$, are located on the *main diagonal*

Note:

- The sum of the diagonal elements of a square matrix A is called the *trace of A*.
- A matrix in which every element is zero, is called a *zero matrix*.

1.2. Matrix Operations

Two matrices $A = [a_{ij}]$ and B= $[b_{ij}]$ are equal, that is, $A = B$, if and only if

$$a_{ij} = b_{ij} \text{ for } i = 1,2, \dots, m \text{ and } j = 1,2, \dots, n \quad (3.2)$$

Two matrices are said to be *conformable for addition /subtraction*, if they are of the same order $m \times n$. Hence, their sum /difference will be another matrix C with the same order as A and B, where each element of C is the sum /difference of the corresponding elements of and, that is,

$$C = A \mp B = [a_{ij} \mp b_{ij}] \quad (3.3)$$

Example 3.2.1

Compute $A + B$ and $A - B$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 2 & 5 \end{bmatrix}$$

Solution

$$A + B = \begin{bmatrix} 1+2 & 2+3 & 3+0 \\ 0+(-1) & 1+2 & 4+5 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 3 \\ -1 & 3 & 9 \end{bmatrix},$$

and

$$A - B = \begin{bmatrix} 1-2 & 2-3 & 3-0 \\ 0-(-1) & 1-2 & 4-5 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 \\ 1 & -1 & -1 \end{bmatrix}$$

Check with MATLAB

Try it! In the command window

```
>> A= [1 2 3;0 1 4]; B= [2 3 0;-1 2 5];      % Define matrices A and B
>> A+B                                         % Add A and B
ans =
     3     5     3
    -1     3     9
>> A-B
ans =
    -1    -1     3
    -1    -1    -1
```

Example 3.2.2

Multiply the matrix $A = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$ by

(a) $k_1 = 5$ and (b) $k_2 = -3 + 2i$

Solution

(a) $k_1.\ A = 5 \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5\times 1 & 5\times(-2) \\ 5\times 2 & 5\times 3 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ 10 & 15 \end{bmatrix}$

(b) $k_2.\ A = (-3+2i) \times \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} (-3+2i)\times 1 & (-3+2i)\times(-2) \\ (-3+2i)\times 2 & (-3+2i)\times 3 \end{bmatrix} =$

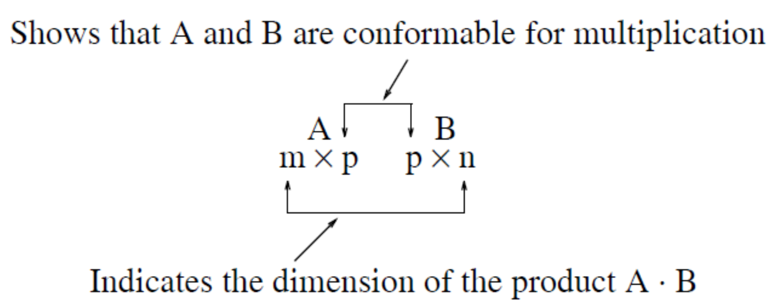
$\begin{bmatrix} -3+2i & 6-4i \\ -6+4i & -9+6i \end{bmatrix}$

Check with MATLAB

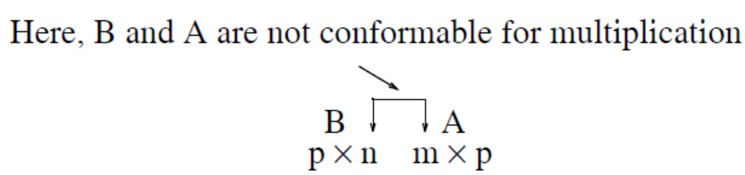
Try it! In the command window

```
>> k1=5; k2= (-3+2*i);        % Define the scalers k1 and k2
>> A=[1 -2; 2 3];             % Define matrix A
>> k1* A                      % Multiply matrix A by constant k1
 ans =
      5       10
      10      15
>> k2* A                      % Multiply matrix A by constant k2
 ans =
      -3.0000 + 2.0000 i       6.0000 - 4.0000 i
       -6.0000+ 4.0000 i       9.0000+ 6.0000 i
```

Two matrices A and B are said to be conformable for multiplication $A.B$ in that order, only when the number of columns of matrix A is equal to the number of rows of matrix B. That is, the product $A.B$ (but not $B.A$) is conformable for multiplication only if A is an $m \times p$ and matrix B is an $p \times n$ matrix. The product will then be $A.B$ an $m \times n$ matrix. A convenient way to determine if two matrices are conformable for multiplication is to write the dimensions of the two matrices side–by–side as shown below.



For the product $B \cdot A$ we have:



For matrix multiplication, the operation is row by column. Thus, to obtain the product $A.B$, we multiply each element of a row of A by the corresponding element of a column of B; then, we add these products.

Example 3.2.3

Given that $C = [2\ 3\ 4]$ and $D = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, Compute the product $C.D$ and $D.C$.

Solution

The dimensions of matrices C and D are respectively $1 \times 3,\ 3\ \times 1$; therefore, the product $C.D$ is feasible, and will result in a 1×1 , that is,

$$C.D = [2\ 3\ 4] \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \left[\ (2 \times 1) + \big(3 \times (-1)\big) + (4 \times 2)\right] = [7]$$

The dimensions for D and C are respectively $3 \times 1, 1 \times 3$ and therefore, the product $D.C$ is also feasible. Multiplication of these will produce a 3 × 3 matrix as follows.

$$D.C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [2\ 3\ 4] = \begin{bmatrix} (1 \times 2) & (1 \times 3) & (1 \times 4) \\ (-1 \times 2) & (-1 \times 3) & (-1 \times 4) \\ (2 \times 2) & (2 \times 3) & (2 \times 4) \end{bmatrix} = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \end{bmatrix}$$

Check with MATLAB

Try it! In the command window

```
>> C= [2 3 4]; D= [1; -1; 2];       % Define matrices C and D
>> C* D                             % Multiply C by D
 ans =
       7
>> D* C                             % Multiply D by C
 ans =
      2      3       4
     -2     -3      -4
      4      6       8
```

1.3. Special forms of Matrices

A square matrix is said to be upper triangular when all the elements below the diagonal are zero. The matrix A below is an upper triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & \ddots & \dots & \dots \\ \dots & \dots & 0 & \ddots & \dots \\ 0 & 0 & 0 & \dots & a_{mn} \end{bmatrix} \quad (3.3)$$

- A square matrix is said to be lower triangular, when all the elements above the diagonal are zero. The matrix B below is a lower triangular matrix.

$$B = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ \dots & \dots & \ddots & 0 & 0 \\ \dots & \dots & \dots & \ddots & 0 \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \quad (3.4)$$

- A square matrix is said to be diagonal, if all elements are zero, except those in the diagonal. The matrix B below is a diagonal matrix.

$$C = \begin{bmatrix} a_{11} & 0 & 0 & \dots & 0 \\ 0 & a_{22} & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \dots & a_{mn} \end{bmatrix} \quad (3.5)$$

- A diagonal matrix is called a *scalar matrix*, if $a_{11} = a_{22} = a_{33} = \cdots = a_{nn} = k$ where k is a scalar. The matrix D below is a scalar matrix with $k = 4$.

$$D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad (3.6)$$

- A scalar matrix with $k = 1$, is called an *identity matrix I*. Shown below are 2×2, 3×3, and 4×4 identity matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3.7)$$

The MATLAB **eye(n)** function displays an $n \times n$ identity matrix. For example,

Try it! In the command window

```
>> eye (4)        % displays 4 by 4 identity matrix
ans =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

The function **eye(size(A))**, produces an *identity* matrix whose size is the same as matrix A. For example, let A be defined as

Try it! In the command window

```
>> A= [1 3 1; -2 1 -5; 4 -7 6]          % Define matrix A
 A =
    1     3     1
   -2     1    -5
    4    -7     6
>> eye (size (A))
ans =
     1     0     0
     0     1     0
     0     0     1
```

- The *transpose* of a matrix A denoted as A^T, is the matrix that is obtained when the rows and columns of matrix A are interchanged. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \qquad (3.8)$$

In MATLAB we use the *apostrophe* (′) symbol to denote and obtain the *transpose* of a matrix. Thus, for the above example,

Try it! In command window

```
>> A= [1 2 3; 4 5 6]        % Define Matrix A
ans =
     1     2     3
     4     5     6
>> A'                       % Display the transpose of A
ans =
     1     4
     2     5
     3     6
```

- A *symmetric* matrix A is one such that $A = A^T$, that is, the transpose of a matrix A is the same as A. An example of a *symmetric* matrix is shown below.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -5 \\ 3 & -5 & 6 \end{bmatrix} = A \qquad (3.9)$$

- If a matrix A has complex numbers as elements, the matrix obtained from A by replacing each element with its conjugate is called the *conjugate* of A, and it is denoted as A^*.

$$A = \begin{bmatrix} 1+j2 & j \\ 3 & 2-j3 \end{bmatrix} \qquad A^* = \begin{bmatrix} 1-j2 & -j \\ 3 & 2+j3 \end{bmatrix} \qquad (3.10)$$

- MATLAB has built−in functions that compute the complex conjugate of a number. The *conj(A)* computes the conjugate of a matrix. Using MATLAB with matrix A defined as above, we obtain

Try it! In the command window

```
>> A= [1+2i i; 3 2-3i]       % Define and display matrix A
 A =
   1.0000 +     2.0000i          0 +           1.0000i
   3.0000                        2.0000 -      3.0000i
>> conj_A= conj (A)           % Compute and display the conjugate of matrix A
 conj_A =
          1.0000 -     2.0000i       0 -            1.0000i
          3.0000                     2.0000 +      3.0000i
```

- A square matrix A such that $A^T = -A$ is called *skew−symmetric*. For example,

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix} = -A \tag{3.11}$$

- A square matrix A such that $A^{T*} = A$, is called *Hermitian*. For example,

$$A = \begin{bmatrix} 1 & 1-j & 2 \\ 1+j & 3 & j \\ 2 & -j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} 1 & 1+j & 2 \\ 1-j & 3 & -j \\ 2 & j & 0 \end{bmatrix} = A \tag{3.12}$$

- A square matrix A such that $A^{T*} = -A$, is called *skew−Hermitian*. For example

$$A = \begin{bmatrix} j & 1-j & 2 \\ -1-j & 3j & j \\ -2 & j & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} j & -1-j & -2 \\ 1-j & 3j & j \\ 2 & j & 0 \end{bmatrix} \quad A^{T*} = \begin{bmatrix} -j & -1+j & -2 \\ 1+j & -3j & -j \\ 2 & -j & 0 \end{bmatrix} = -A \tag{3.13}$$

1.4. Determinants

Let matrix A be defined as the square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix} \tag{3.14}$$

then, the *determinant of A*, denoted as $detA$, is defined as

$$\begin{aligned} \det A = {} & a_{11}a_{22}a_{33}\dots a_{nn} + a_{12}a_{23}a_{34}\dots a_{n1} + a_{13}a_{24}a_{35}\dots a_{n2} + \dots \\ & -a_{n1}\dots a_{22}a_{13}\dots -a_{n2}\dots a_{23}a_{14} - a_{n3}\dots a_{24}a_{15} - \dots \end{aligned} \tag{3.15}$$

The determinant of a square matrix of order *n* is referred to as *determinant of order n.*

Let $detA$ be a *determinant of order 2* for matrix A, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{3.16}$$

Then

$$detA = a_{11}a_{22} - a_{21}a_{12} \tag{3.17}$$

Example 3.4.1

Given that $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix}$, then compute the $detA$ and $detB$.

Solution

$$detA = 1 \times 4 - 3 \times 2 = 4 - 6 = -2$$

$$detB = 2 \times 0 - 2 \times (-1) = 0 - (-2) = 2$$

Check with MATLAB

Try it! In command window

```
>> A= [1 2; 3 4]; B= [2 -1; 2 0];      % Define matrices A and B
>> det (A)                             % compute determinant of A
 ans =
       -2
>> det (B)                             % compute determinant of B
ans =
       2
```

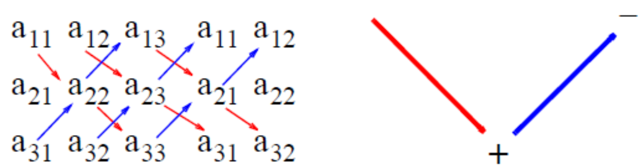
Now let A be a matrix of order 3, that is,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{3.18}$$

Then $detA$, is found from

$$\det A = a_{11}\,a_{22}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{22}\,a_{33} \quad (3.19)$$

A convenient method to evaluate the determinant of order 3, is to write the first two columns to the right of the matrix of order 3×3, and add the products formed by the diagonals from upper left to lower right; then subtract the products formed by the diagonals from lower left to the upper right as shown on the diagram below. When this is done properly, we obtain (3.19) above.



This method works only with second and third order determinants. To evaluate higher order determinants, we must first compute the *cofactors*; these will be defined shortly.

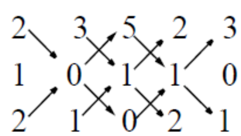
Example 3.4.2

Compute the $detA$ and $detB$ where A and B are given by

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -3 & -4 \\ 1 & 0 & -2 \\ 0 & -5 & -6 \end{bmatrix}$$

Solution

detA =

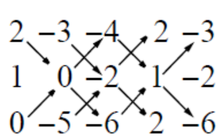


Or

$$\det A = (2 \times 0 \times 0) + (3 \times 1 \times 1) + (5 \times 1 \times 1) - (2 \times 0 \times 5) - (1 \times 1 \times 2) - (0 \times 1 \times 3) = 11 - 2 = 9$$

Likewise,

detB =



Or

$$\det B = [2\times 0\times(-6)] + [(-3)\times(-2)\times 0] + [(-4)\times 1\times(-5)] - [0\times 0\times(-4)] - [(-5)\times(-2)\times 2] - [(-6)\times 1\times(-3)] = 20 - 38 = -18$$

Check with MATLAB

Try it! In command window

```
>> A= [2 3 5; 1 0 -1; 2 1 0]; det (A)      % Define matrix A and compute detA
ans =
     9
>> B= [2 -3 -4; 1 0 -2; 0 -5 -6]; det (B)      % Define matrix B and compute detB
ans =
     -18
```

1.5. Minors and cofactors

Recall the matrix A defined in equation (3.14).

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

If we remove the elements of its $i-th$ row, and $j-th$ column, the determinant of the remaining $n-1$ square matrix is called the *minor of determinant A*, and it is denoted as $[M_{ij}]$. The signed minor $(-1)^{i+j}.\ [M_{ij}]$ is called the *cofactor* of a_{ij} and it is denoted as α_{ij}.

Example 3.5.1

Given that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Compute minors $[M_{11}]$, $[M_{12}]$, $[M_{13}]$ and cofactors α_{11}, α_{12}, α_{13}.

Solution

$$[M_{11}] = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \qquad [M_{12}] = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \qquad [M_{11}] = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

and

$$\alpha_{11} = (-1)^{1+1}[M_{11}] = [M_{11}] \qquad \alpha_{12} = (-1)^{1+2}[M_{12}] = -[M_{12}] \qquad \alpha_{13} = [M_{13}] = (-1)^{1+3}[M_{13}]$$

The remaining minors $[M_{21}]$, $[M_{22}]$, $[M_{23}]$, $[M_{31}]$, $[M_{32}]$, $[M_{33}]$

And cofactors α_{21}, α_{22}, α_{23}, α_{31}, α_{32}, α_{33} are defined similarly. (check that!)

Example 3.5.2

Compute the cofactors of the following matrix, $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$

Solution

$$\alpha_{11} = (-1)^{1+1}\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} = 20 \qquad \alpha_{12} = (-1)^{1+2}\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} = 10$$

$$\alpha_{13} = (-1)^{1+3}\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 0 \qquad \alpha_{21} = (-1)^{2+1}\begin{bmatrix} 2 & -3 \\ 2 & -6 \end{bmatrix} = 6$$

$$\alpha_{22} = (-1)^{2+2}\begin{bmatrix} 1 & -3 \\ -1 & -6 \end{bmatrix} = -9 \qquad \alpha_{23} = (-1)^{2+3}\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix} = -4$$

$$\alpha_{31} = (-1)^{3+1}\begin{bmatrix} 2 & -3 \\ -4 & 2 \end{bmatrix} = -8, \qquad \alpha_{32} = (-1)^{3+2}\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} = -8$$

$$\alpha_{33} = (-1)^{3+3}\begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -8$$

It is useful to remember that the signs of the cofactors follow the pattern

$$\begin{matrix} + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \\ - & + & - & + & - \\ + & - & + & - & + \end{matrix}$$

that is, the cofactors on the diagonals have the same sign as their minors.

Let A be a square matrix of any size; the value of the determinant of A is the sum of the products obtained by multiplying each element of *any* row or *any* column by its cofactor.

Example 3.5.3

Compute the determinant of A using the elements of the first row. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & -4 & 2 \\ -1 & 2 & -6 \end{bmatrix}$

Solution

$$\det A = 1\begin{bmatrix} -4 & 2 \\ 2 & -6 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -1 & -6 \end{bmatrix} - 3\begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} = 1 \times 20 - 2 \times (-10) - 3 \times 0 = 40$$

Check with MATLAB

Try it! In command window

```
>> A= [1 2 -3; 2 -4 2; -1 2 -6]; det (A)      % Define matrix A and compute detA
ans =
     40
```

To determine the determinant of a 4×4 matrix, see the following:

The fourth-order determinant can first be expressed as the sum of the products of the elements of its first row by its cofactor as shown below.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$+ a_{31}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{41}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}$$

Determinants of order five or higher can be evaluated similarly.

Example 3.5.4

Compute the determinant of the following matrix, $A = \begin{bmatrix} 2 & -1 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 4 & 0 & 3 & -2 \\ -3 & 0 & 0 & 1 \end{bmatrix}$

Solution

$$A = \underbrace{2\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[a]} \underbrace{-(-1)\begin{bmatrix} -1 & 0 & -3 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}}_{[b]} \underbrace{+4\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}}_{[c]} \underbrace{-(-3)\begin{bmatrix} -1 & 0 & -3 \\ 1 & 0 & -1 \\ 0 & 3 & -2 \end{bmatrix}}_{[d]}$$

Next, using the procedure of Example 3.5.2 or example 3.5.3, we find

$$[a] = 6, \ \ [b] = -3, \ \ [c] = 0, \ \ [d] = -36$$

Thus, $detA = [a] + [b] + [c] + [d] = 6 + (-3) + 0 + (-36) = -33$

Check with MATLAB

Try it! In command window

```
>> A= [2 -1 0 -3; -1 1 0 -1; 4 0 3 -2; -3 0 0 1]; delta = det (A)
delta =
     -33
```

To compute the determinant of a $n \times n$ matrix MATLAB users may use the following code.

Try it! In Editor window

```
% This file computes the determinant of a nxn matrix
% It must be saved as function (user defined) file
% detnxn.m in the current Work Directory. Make sure
% that his directory is added to MATLAB's search
% path accessed from the Editor Window as File>Set Path>
% Add Folder. It is highly recommended that this
% function file is created in MATLAB's Editor Window.
function y=detnxn(A);
% The following statement initializes y
y = 0;
% The following statement defines the size of the matrix A
[n, n] =size(A);
% MATLAB allows us to use the user-defined functions to be recursively
% called on themselves so we can call det2x2(A) for a 2x2 matrix,
% and det3x3(A) for a 3x3 matrix.
if n==2
y=det2x2(A);
return
end
%
if n==3
y=det3x3(A);
return
end
% For 4x4 or higher order matrices we use the following:
% (We can define n and matrix A in Command Window
for i=1: n
y=y+(-1) ^(i+1) *A (1, i) *detnxn (A (2: n, [1:(i-1) (i+1): n]));
end
% To run this program, define the nxn matrix in
% MATLAB's Command Window as A= [....] and then
% type detnxn(A) at the command prompt.
```

After all the aforementioned information about the determinants, the following properties need to be considered:

Property 1:

- If all elements of one row or one column are zero, the determinant is zero. An example of this is the determinant of the cofactor [c] above.

Property 2:

- If all the elements of one row or column are m times the corresponding elements of another row or column, the determinant is zero. For example, if

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \text{then } A = \begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} 2 & 4 \\ 3 & 6 \\ 1 & 2 \end{matrix} = 12 + 4 + 6 - 6 - 4 + 12 = 0$$

Here, $detA$ is zero because the second column in A is 2 times the first column. Check with MATLAB.

Try it! In command window

```
>> A = [2 4 1; 3 6 1; 1 2 1]; det(A)
 Ans =
       0
```

Property 3:

- If two rows or two columns of a matrix are identical, the determinant is zero. This follows from Property 2 with $m = 1$.

After absorbing the idea of using MATLAB to solve math problems, we proceed to deal with some important concept that is, solving a system of linear equations. Regarding this subject, we will discuss some basic numerical methods, which are Cramer's rule, the Gaussian elimination method and simultaneous equations.

1.6. Cramer's Rule

Consider the system of the three equations below

$$\begin{aligned} a_{11}x + \quad a_{12}y + \quad a_{13}z = \quad A \\ a_{21}x + \quad a_{22}y + \quad a_{23}z = \quad B \\ a_{31}x + \quad a_{32}y + \quad a_{33}z = \quad C \end{aligned} \tag{3.20}$$

And let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \; D_1 = \begin{vmatrix} A & a_{12} & a_{13} \\ B & a_{22} & a_{23} \\ C & a_{32} & a_{33} \end{vmatrix}, D_2 = \begin{vmatrix} a_{11} & A & a_{13} \\ a_{21} & B & a_{23} \\ a_{31} & C & a_{33} \end{vmatrix}, D_3 = \begin{vmatrix} a_{11} & a_{12} & A \\ a_{21} & a_{22} & B \\ a_{31} & a_{32} & C \end{vmatrix}$$

Cramer's rule states that the unknowns $x, y,$ and z can be found from the relations

$$\boldsymbol{x} = \frac{\boldsymbol{D_1}}{\Delta}, \quad \boldsymbol{y} = \frac{\boldsymbol{D_2}}{\Delta}, \quad \boldsymbol{z} = \frac{\boldsymbol{D_3}}{\Delta} \tag{3.21}$$

- provided that the determinant Δ (delta) is not zero.

- We observe that the numerators of equation (3.21) are formed from Δ by substituting the known values $A, B,$ and $C,$ for the coefficients of the desired unknown.
- Cramer's rule applies to systems of two or more equations.
- If equation (3.20) is a homogeneous set of equations, if $A = B + C,$ then $D1, D2, D3$ are all zero as we found in Property 1 above. Then, $x = y = z = 0$ also.

Example 3.6.1

Use Cramer's rule to find v_1, v_2 and v_3 if

$$\begin{array}{cccc} 2v_1 - 5 & -v_2 + & 3v_3 = & 0 \\ -2v_3 & -3v_2 & -4v_1 = & 8 \\ v_2 + & 3v_1 - 4 & -v_3 = & 0 \end{array}$$

Verify your answers with MATLAB.

Solution

Rearranging the unknowns v, and transferring known values to the right side, we obtain

$$\begin{array}{cccc} 2v_1 & -v_2 + & 3v_3 = & 5 \\ -4v_1 & -3v_2 & -2v_3 = & 8 \\ 3v_1 + & v_2 & -v_3 = & 4 \end{array}$$

Now by Cramer's rule, we have

$$\Delta = \left|\begin{array}{ccc|cc} 2 & -1 & 3 & 2 & -1 \\ -4 & -3 & -2 & -4 & -3 \\ 3 & 1 & -1 & 3 & 1 \end{array}\right. = 6 + 6 - 12 + 27 + 4 + 4 = 35$$

$$D_1 = \left|\begin{array}{ccc|cc} 5 & -1 & 3 & 5 & -1 \\ 8 & -3 & -2 & 8 & -3 \\ 4 & 1 & -1 & 4 & 1 \end{array}\right. = 15 + 8 + 24 + 36 + 10 - 8 = 85$$

$$D_2 = \left|\begin{array}{ccc|cc} 2 & 5 & 3 & 2 & 5 \\ -4 & 8 & -2 & -4 & 8 \\ 3 & 4 & -1 & 3 & 4 \end{array}\right. = -16 - 30 - 48 - 72 + 16 - 20 = 170$$

$$D_3 = \left|\begin{array}{ccc|cc} 2 & -1 & 5 & 2 & -1 \\ -4 & -3 & 8 & -4 & -3 \\ 3 & 1 & 4 & 3 & 1 \end{array}\right. = -24 - 24 - 20 + 45 - 16 - 16 = -55$$

Using equation (3.21) we obtain

$$v_1 = \frac{D_1}{\Delta} = \frac{85}{35} = \frac{17}{7}, \quad v_2 = \frac{D_2}{\Delta} = \frac{170}{35} = \frac{34}{7}, \quad v_3 = \frac{D_3}{\Delta} = \frac{-55}{35} = \frac{-11}{7}$$

Now, verify with MATLAB as follows.

Try it! In Editor window

```
% The following script will compute and display the values of v1, v2 and v3.
format rat                        % Express answers in ratio form
B=[2 -1 3; -4 -3 -2; 3 1 -1];    % The elements of the determinant D
delta=det(B);                     % Compute the determinant D of B
d1=[5 -1 3; 8 -3 -2; 4 1 -1];    % The elements of D1
detd1=det(d1);                    % Compute the determinant of D1
d2=[2 5 3; -4 8 -2; 3 4 -1];     % The elements of D2
detd2=det(d2);                    % Compute the determinant of D2
d3=[2 -1 5; -4 -3 8; 3 1 4];     % The elements of D3
detd3=det(d3);                    % Compute he determinant of D3
v1=detd1/delta;                   % Compute the value of v1
v2=detd2/delta;                   % Compute the value of v2
v3=detd3/delta;                   % Compute the value of v3
disp('v1='); disp (v1);           % Display the value of v1
disp('v2='); disp (v2);           % Display the value of v2
disp('v3='); disp (v3);           % Display the value of v3
```

In Command window you will see,

```
v1=
   17/7
v2=
   -34/17
v3=

   -11/7
```

1.7. Gaussian Elimination Method

We can find the unknowns in a system of two or more equations also by the Gaussian elimination method. With this method, the objective is to eliminate one unknown at a time. This can be done by multiplying the terms of any of the equations of the system by a number such that we can add (or subtract) this equation to another equation in the system so that one of the unknowns will be eliminated. Then, by substitution to another equation with two unknowns, we can find the second unknown. Subsequently, the substitution of the two values found can be made into an equation with three unknowns from which we can find the value of the third unknown. This procedure is repeated until all unknowns are found. This method is best illustrated with the following example which consists of the same equations as the previous example.

Example 3.7.1

Use Gaussian elimination method to find v_1, v_2 and v_3 of

$$\begin{array}{llll} 2v_1 & -v_2 + & 3v_3 = & 5 \\ -4v_1 & -3v_2 & -2v_3 = & 8 \\ 3v_1 + & v_2 & -v_3 = & 4 \end{array} \tag{3.22}$$

Solution

As a first step, we add the first equation of (3.22) with the third to eliminate the unknown v_2 , and we obtain the following equation.

$$5v_1 + 2v_3 = 9 \tag{3.23}$$

Next, we multiply the third equation of (3.22) by 3, and we add it with the second to eliminate v_2. Then, we obtain the following equation

$$5v_1 - 5v_3 = 20 \tag{3.24}$$

Subtraction of (3.24) from (3.23) yields

$$7v_3 = -11 \text{ or } v_3 = \frac{-11}{7} \tag{3.25}$$

Now, we can find the unknown v_1 from either (3.23) or (3.24). By substitution of (3.25) into (3.23), we obtain

$$5v_1 + 2 . \left(\frac{-11}{7}\right) = 9 \text{ or } v_1 = \frac{17}{7} \tag{3.26}$$

Finally, we can find the last unknown v_2 from any of the three equations of (3.22). By substitution into the first equation we obtain

$$v_2 = 2v_1 + 3v_3 - 5 = \frac{34}{7} - \frac{33}{7} - \frac{35}{7} = \frac{-34}{7} \tag{3.27}$$

- These are the same values as those we found in Example 3.6.1

Before we discuss the simultaneous equations, we should know the following concepts.

1.8. The Adjoint of a Matrix

Let us assume that A is an n square matrix and α_{ij} is the cofactor of a_{ij}. Then the adjoint of A, denoted as $adjA$, is defined as the square matrix shown below,

$$\mathrm{adjA} = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} & \dots & \alpha_{n1} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} & \dots & \alpha_{n2} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} & \dots & \alpha_{n3} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{1n} & \alpha_{2n} & \alpha_{3n} & \dots & \alpha_{nn} \end{bmatrix} \tag{3.28}$$

We observe that the cofactors of the elements of the $i-th$ row (column) of A, are the elements of the $i-th$ column (row) of $adjA$.

Example 3.8.1

Compute the $adjA$ given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$$

Solution

$$adjA = \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

1.9. Singular and Non-Singular Matrices

An n square matrix A is called singular if $detA = 0$; if $detA \neq 0$, A is called non−singular. If an n square matrix A is nearly singular, that is, if the determinant of that matrix is very small, the matrix is said to be ill−conditioned see [1].

Example 3.9.1

Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$, determine whether this matrix is singular or non-singular.

Solution

$$detA = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{vmatrix} \begin{matrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \end{matrix} = 21 + 24 + 30 - 27 - 20 - 28 = 0$$

Therefore, matrix A is singular.

1.10. The Inverse of a Matrix

If A and B are square matrices such that $AB = BA = I$, where I is the identity matrix, B is called the inverse of A, denoted as $B = A^{-1}$, and likewise, A is called the inverse of B, that is, $A = B^{-1}$

If a matrix A is non−singular, we can compute its inverse from the following relation

$$A^{-1} = \frac{1}{detA}\, adjA \qquad (3.29)$$

Example 3.10.1

Given that, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}$, Compute the inverse of A.

Solution

Here, $detA = 9 + 8 + 12 - 9 - 16 - 6 = -2$, and since this is a non-zero value, it is possible to compute the inverse of A using (3.29). From Example 3.8.1, we already now that,

$$adjA = \begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix},$$

Hence,

$$A^{-1} = \frac{1}{detA}\, adjA = \frac{1}{-2}\begin{bmatrix} -7 & 6 & -1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 3.5 & -3 & 0.5 \\ -0.5 & 0 & 0.5 \\ -0.5 & 0 & -0.5 \end{bmatrix}$$

Check with MATLAB

Try it! In command window

```
>> A= [1 2 3; 1 3 4; 1 4 3], invA=inv(A)        % Define matrix A and compute its inverse
 A =
     1     2     3
     1     3     4
     1     4     3
 inv A=

       3. 5000     -3. 0000     0. 0000

      -0. 5000            0     0. 5000

      -0. 5000      1. 0000    -0. 5000
```

Multiplying a matrix A by its inverse A^{-1} produces the identity matrix I, that is,

$$AA^{-1} = I \text{ or } A^{-1}A = I \tag{3.30}$$

Example 3.10.2

Prove the validity of equation (3.30) for following matrix $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

Proof

$$detA = 8 - 6 = 2 \text{ and } adjA = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{detA} adjA = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix}$$

And

$$AA^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3/2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.11. Solution of Simultaneous equations with Matrices

Consider the relation

$$AX = B \tag{3.31}$$

where A and B are matrices whose elements are known, and X is a matrix (a column vector) whose elements are the unknowns. We assume that A and X are conformable for multiplication. Multiplication of both sides of (3.31) by yields:

$$A^{-1}AX = A^{-1}B = IX = A^{-1}B \tag{3.31}$$

or

$$X = A^{-1}B \tag{3.32}$$

Therefore, we can use (3.32) to solve any set of simultaneous equations that have solutions. We will refer to this method as the *inverse matrix method of solution* of simultaneous equations.

Example 3.11.1

Given the system of equations $\begin{Bmatrix} 2x_1 & +3x_2 & +x_3 & =9 \\ x_1 & +2x_2 & +3x_3 & =6 \\ 3x_1 & +x_2 & +2x_2 & =8 \end{Bmatrix}$, compute the unknowns x_1, x_2, and $x_3$using the inverse matrix method

Solution

In matrix form, the given set of equations is $AX = B$ where

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Then

$$X = A^{-1}B$$

Or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Next, we find the determinant $detA$, and the $adjA$.

$$detA = 18 \quad \text{and} \quad adjA = \begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

Thus,

$$A^{-1} = \frac{1}{detA}\, adjA = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}$$

and by equation (4.53) we obtain the solution as follows.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 1 & -5 & 7 \\ 7 & 1 & -5 \\ -5 & 7 & 1 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 35 \\ 29 \\ 5 \end{bmatrix} = \begin{bmatrix} 35/18 \\ 29/18 \\ 5/18 \end{bmatrix} = \begin{bmatrix} 1.94 \\ 1.61 \\ 0.28 \end{bmatrix}$$

To verify our results, we could use the MATLAB inv(A) function, and multiply $A^{-1}$by B. However, it is easier to use the *matrix left division* operation $X = A \setminus B$; this is MATLAB's solution of$A^{-1}B$ for the matrix equation $AX = B$, where matrix X is the same size as matrix B, see [1].

Check with MATLAB

Try it! In command window

```
>> A= [2 3 1; 1 2 3; 3 1 2]; B = [9; 6; 8]; X= A\ B  % Observe that B is column vector
 X =
    1.9444
    1.6111
    0.2778
```

Example 3.11.2

Consider that electric circuit is shown in the following figure

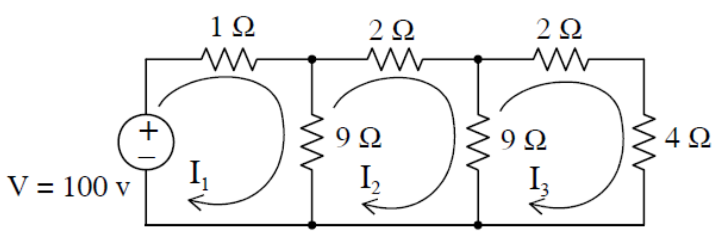


Figure 3.11.1: Electric circuit for example 3.11.2

The mesh equations are given as

$$\begin{aligned} 10I_1 - 9I_2 \qquad &= 100 \\ -9I_1 + 20I_2 - 9I_3 &= 0 \\ -9I_2 + 15I_3 &= 0 \end{aligned}$$

Use the inverse matrix method to compute the values of the currents I_1, I_2 and I_3.

Solution

For this example, the matrix equation is $RI = V$ or $I = R^{-1}V$, where

$$R = \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -9 \\ 0 & -9 & 15 \end{bmatrix}, \quad V = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \quad I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

The next step is to find R^{-1}. This is found from the relation

$$R^{-1} = \frac{1}{detR} adjR$$

Therefore, we find the determinant and the adjoint of R. For this example, we find that

$$detR = 975, \quad adjR = \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}, \text{ then}$$

$$R^{-1} = \frac{1}{detR} adjR = \frac{1}{975} \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}, \text{ and hence,}$$

$$I = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \frac{1}{975} \begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{100}{975} \begin{bmatrix} 219 \\ 135 \\ 81 \end{bmatrix} = \begin{bmatrix} 22.46 \\ 13.85 \\ 8.31 \end{bmatrix}$$

Type equation here.

Check with MATLAB

Try it! In command window

```
>> R= [10 -9 0; -9 20 -9; 0 -9 15]; V = [100; 0; 0]; I= R\ V  % Observe that B is column vector
 I =
     22.4615
     13.8562
     8.31077
```

1.12. Exercise

For Exercises 1 through 3 below, the matrices A,B, and C are defined as:

$$A = \begin{bmatrix} 1 & -1 & -4 \\ 5 & 7 & -2 \\ 3 & -5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 9 & -3 \\ -2 & 8 & 2 \\ 7 & -4 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 6 \\ -3 & 8 \\ 5 & -2 \end{bmatrix}, \text{ and } D = \begin{bmatrix} 1 & -2 & 3 \\ -3 & 6 & -4 \end{bmatrix}$$

1. Perform the following computations, if possible. Verify your answers with MATLAB.

a) A+B b) A+C c) B+D d) C+D e) A-B f) A-C g) B-D h) C-D

2. Perform the following computations, if possible. Verify your answers with MATLAB.

a) A. B b) A.C c) B.D d) C.D e) B.A f) C.A g) D.A h) D.C

3. Perform the following computations, if possible. Verify your answers with MATLAB.

a) $detA$ b) $detB$ c) $detC$ d) $detD$ e) $det(A.B)$ f) $det(A.C)$

4. Solve the following system of equations using Cramer's rule. Verify your answers with MATLAB.

$$\begin{array}{cccc} x_1 & -2x_2 & +x_3 = & -4 \\ -2x_1 & +3x_2 & +x_3 = & 9 \\ 3x_1 & +4x_2 & -5x_3 = & 0 \end{array}$$

5. Repeat Exercise 4 using the Gaussian elimination method.

6. Use the MATLAB **det(A)** function to find the unknowns of the system of equations below.

$$\begin{array}{ccccc} -x_1 & +2x_2 & -3x_3 & +5x_4 & = 14 \\ x_1 & +3x_2 & +2x_3 & -x_4 & = 9 \\ 3x_1 & -3x_2 & +2x_3 & +4x_4 & = 19 \\ 4x_1 & +2x_2 & +5x_3 & +x_4 & = 27 \end{array}$$

Chapter II: Least Squares Regression

Motivation

Often in physics and engineering coursework, we are asked to determine the state of a system given the parameters of the system. For example, the relationship between the force exerted by a linear spring, F, and the displacement of the spring from its natural length, x, is usually represented by the model,

$$F = kx,$$

where k is the spring stiffness. We are then asked to compute the force for a given k and x value. However, in practice, the stiffness and in general, most of the parameters of a system, are not known *a priori*. Instead, we are usually presented with data points about how the system has behaved in the past. For our spring example, we may be given (x, F) data pairs that have been previously recorded from an experiment. Ideally, all these data points would lie exactly on a line going through the origin (since there is no force at zero displacement). We could then measure the slope of this line and get our stiffness value for k. However, practical data usually has some measurement noise because of sensor inaccuracy, measurement error, or a variety of other reasons. Figure 2.1 shows an example of what data might look like for a simple spring experiment. This chapter teaches methods of finding the "most likely" model parameters given a set of data; for example, how to find the spring stiffness in our mock experiment. By the end of this chapter you should understand how these methods choose model parameters, the importance of choosing the correct model, and how to implement these methods in MATLAB.

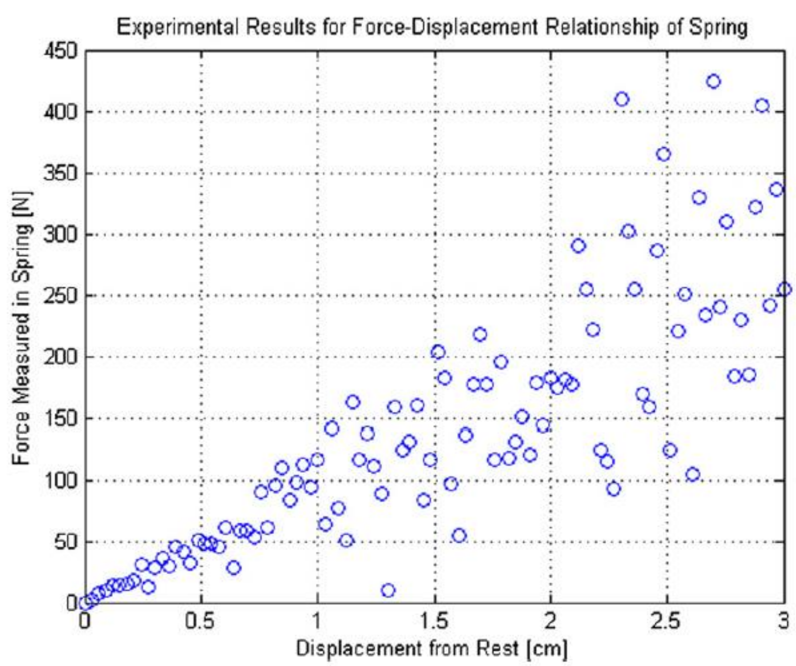


Fig2.1: *Results from force-displacement experiment for spring (fictional). The theoretical linear relationship between force and displacement in a linear spring is F = kx. What do you think k should be given as the experimental data?*

2.1 Least Squares Regression Problem Statement

Given a set of independent data points x_i and dependent data points $y_i,\ i = 1, \ldots, m$, we would like to find an **estimation function**, $\hat{y}(x)$, that describes the data as well as possible. Note that $\hat{y}$ can be a function of several variables, but for the sake of this discussion, we restrict the domain of $\hat{y}$to be a single variable. In least squares regression, the estimation function must be a linear combination of **basis functions**, $f_i(x)$. That is, the estimation function must be of the form

$$\hat{y}(x) = \sum_{i=1}^{n} \alpha_i f_i(x)$$

The scalars α_i are referred to as the **parameters** of the estimation function, and each basis function must be linearly independent from the others. In other words, in the proper "functional space" no basis function should be expressible as a linear combination of the other functions. Note: In general, there are significantly more data points, *m*, than basis functions, *n* (i.e., *m* >> *n*).

TRY IT! Create an estimation function for the force-displacement relationship of a linear spring. Identify the basis function(s) and model parameters. The relationship between the force, *F*, and the displacement, *x*, can be described by the function $F = kx$ where k is the spring stiffness. The only basis function is the function $f_i(x) = x$ and the model parameter to find is $\alpha_1 = k$.

The goal of **least squares regression** is to find the parameters of the estimation function that minimize the **total squared error**, *E*, defined by

$$E = \sum_{i=1}^{m} (\hat{y} - y_i)^2.$$

The **individual errors** or **residuals** are defined as

$$e_i = (\hat{y} - y_i)\ .$$

If *e* is the vector containing all the individual errors, then we are also trying to minimize

$$E = \|e\|_2^2\ ,$$

which is the L_2 norm defined in the previous chapter. In the next two sections we derive the least squares method of finding the desired parameters. The first derivation comes from linear algebra, and the second derivation comes from multivariable calculus. Although they are different derivations, they lead to the same least squares formula. You are free to focus on the section with which you are most comfortable.

2.2 Least Squares Regression Derivation (Linear Algebra)

First, we enumerate the estimation of the data at each data point x_i.

$$\hat{y}(x_1) = \alpha_1\, f_1(x_1) + \alpha_2\, f_2(x_1) + \cdots + \alpha_n\, f_n(x_1)$$
$$\hat{y}(x_2) = \alpha_1\, f_1(x_2) + \alpha_2\, f_2(x_2) + \cdots + \alpha_n\, f_n(x_2)$$
$$\cdots$$
$$\hat{y}(x_m) = \alpha_1\, f_1(x_m) + \alpha_2\, f_2(x_m) + \cdots + \alpha_n\, f_n(x_m)$$

Let $X \in \mathbb{R}^n$ be a column vector such that the i -th element of X contains the value of the i -th x-data point, x_1 , $\hat{Y}$ be a column vector with elements, $\hat{Y}_i = \hat{y}(x_i)$, β be a column vector such that $\beta_i = \alpha_i$, $F_i(x)$ be a function that returns a column vector of $f_i(x)$ computed on every element of x, and A be an $m \times n$ matrix such that the i -th column of A is $F_i(x)$. Given this notation, the previous system of equations becomes $\hat{Y} = A\beta$.
Now if Y is a column vector such that $Y_i = y_i$, the total squared error is given by

$$E = \left\|\hat{Y} - Y\right\|_2^2 \ .$$

You can verify this by substituting the definition of the L_2 norm. Since we want to make E as small as possible and norms are a measure of distance, this previous expression is equivalent to saying that we want $\hat{Y}$ and Y to be a "close as possible." Note that in general Y will not be in the range of A and therefore $E > 0$.
Consider the following simplified depiction of the range of A; see Figure 13.2. Note this is *not* a plot of the data points (x_i , y_i). From observation, the vector in the range of A, $\hat{Y}$, that is closest to Y is the one that can point perpendicularly to Y. Therefore, we want a vector $Y - \hat{Y}$ that is perpendicular to the vector $\hat{Y}$. Recall from the chapter on Linear Algebra that two vectors are perpendicular, or orthogonal, if their dot product is 0. Noting that the dot product between two vectors, v and w, can be written as

$dot(v, w) = v^T w$, we can state that $\hat{Y}$ and $Y - \hat{Y}$ are perpendicular if

$$dot\left(\hat{Y}, Y - \hat{Y}\right) = 0;$$

therefore, $\hat{Y}^T\left(Y - \hat{Y}\right) = 0$, which is equivalent to $(A\beta)^T(Y - A\beta) = 0$.

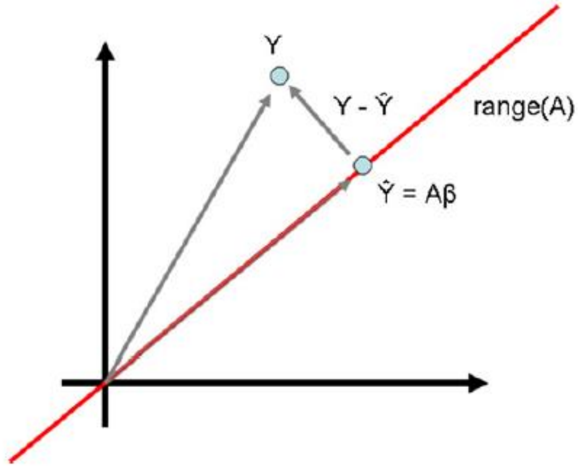


Fig2.2: *Illustration of the L2 projection of Y on the range of A.*

Noting that for two matrices A and B, $(AB)^T = B^T A^T$ and using distributive properties of vector multiplication, this is equivalent to

$$\beta^T A^T Y - \beta^T A^T A\beta = \beta^T (A^T Y - A^T A\beta) = 0.$$

The solution, $\beta = \mathbf{0}$, is a trivial solution, so we use $A^T Y - A^T A \beta = 0$ to find a more interesting solution. Solving this equation for β gives the **least squares regression formula**:

$$\beta = (A^T A)^{-1} A^T Y$$

Note that $(A^T A)^{-1} A^T$ is called the **pseudo-inverse** of A and exists when $m > n$ and A has linearly independent columns.

2.3 Least Squares Regression Derivation (Multivariable Calculus)

Recall that the total error for m data points and n basis functions is:

$$E = \sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} (\hat{y}(x_i) - y_i)^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_j \; f_j(x_i) - y_i \right)^2$$

which is an n-dimensional paraboloid in α_k . From calculus, we know that the minimum of a paraboloid is where all the partial derivatives equal zero. So, taking partial derivative of E with respect to the variable α_k (remember that in this case the parameters are our variables), setting the system of equations equal to 0 and solving for the α_k 's should give the correct results. The partial derivative with respect to α_k and setting equal to 0 yields:

$$\frac{\partial E}{\partial \alpha_i} = \sum_{i=1}^{m} 2\left(\sum_{j=1}^{n} \alpha_j\ f_j(x_i) - y_i\right) f_k(x_i) = 0$$

With some rearrangement, the previous expression can be manipulated to the following:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_j\, f_j(x_i) f_k(x_i) - \sum_{i=1}^{m} y_i\, f_k(x_i) = 0$$

and further rearrangement taking advantage of the fact that addition commutes result in:

$$\sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} f_j(x_i) f_k(x_i) = \sum_{i=1}^{m} y_i\, f_k(x_i)$$

Now let X be a column vector such that the i -th element of X is x_i and Y similarly constructed, and let $F_j(X)$ be a column vector such that the i -th element of $F_j(X)$ is $f_j(x_i)$. Using this notation, the previous expression can be rewritten in vector notation as:

$$\left[F_k^T(X)F_1(X),\, F_k^T(X)F_2(X), \ldots, F_k^T(X)F_j(X), \ldots, F_k^T(X)F_n(X)\right] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_j \\ \cdots \\ \alpha_n \end{bmatrix} = F_k^T(X)Y.$$

If we repeat this equation for every k, we get the following system of linear equations in matrix form:

$$\begin{bmatrix} F_1^T(X)F_1(X), F_1^T(X)F_2(X), \ldots, F_1^T(X)F_j(X), \ldots, F_1^T(X)F_n(X) & \\ F_2^T(X)F_1(X), F_2^T(X)F_2(X), \ldots, F_2^T(X)F_j(X), \ldots, F_2^T(X)F_n(X) & \\ & \cdots\ \cdots \\ F_n^T(X)F_1(X), F_n^T(X)F_2(X), \ldots, F_n^T(X)F_j(X), \ldots, F_n^T(X)F_n(X) & \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdots \\ \alpha_j \\ \cdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} F_1^T(X)Y \\ F_2^T(X)Y \\ \cdots \\ F_n^T(X)Y \end{bmatrix}.$$

If we let $A = [F_1(X), F_2(X), \ldots, F_j\ (X), \ldots, F_n(X)]$ and β be a column vector such that j -th element of β is α_j, then the previous system of equations becomes

$$A^T\, A\, \beta = A^T\, Y\ ,$$

and solving this matrix equation for β gives $\beta = (A^T A)^{-1} A^T Y$, which is exactly the same formula as the previous derivation.

2.4 Least Squares Regression in MATLAB

Recall that if we enumerate the estimation of the data at each data point, x_i , this gives us the following system of equations:

$$\hat{y}(x_1) = \alpha_1 f_1(x_1) + \alpha_2 f_2(x_1) + \cdots + \alpha_n f_n(x_1)$$
$$\hat{y}(x_2) = \alpha_1 f_1(x_2) + \alpha_2 f_2(x_2) + \cdots + \alpha_n f_n(x_2)$$
$$\cdots$$
$$\hat{y}(x_m) = \alpha_1 f_1(x_m) + \alpha_2 f_2(x_m) + \cdots + \alpha_n f_n(x_m)$$

If the data was absolutely perfect (i.e., no noise), then the estimation function would go through all the data points, resulting in the following system of equations:

$$y_1 = \alpha_1 f_1(x_1) + \alpha_2 f_2(x_1) + \cdots + \alpha_n f_n(x_1)$$
$$y_2 = \alpha_1 f_1(x_2) + \alpha_2 f_2(x_2) + \cdots + \alpha_n f_n(x_2)$$
$$\cdots$$
$$y_m = \alpha_1 f_1(x_m) + \alpha_2 f_2(x_m) + \cdots + \alpha_n f_n(x_m)$$

If we take *A* to be as defined previously, this would result in the matrix equation

$$y = Ax$$

However, since the data is not perfect, there will not be an estimation function that can go through all the data points, and this system will have *no solution*. However, recall in the previous chapter that

x =A\y

would return a solution even if no solution existed. It turns out the solution returned by MATLAB for this command is the least squares solution derived in the previous two sections. In other words, if there is a solution, x = A\y will return one, otherwise it will return the *x* that is the closest to being a solution to the matrix equation. The pseudo-inverse for of *A* can be computed using the MATLAB function pinv.

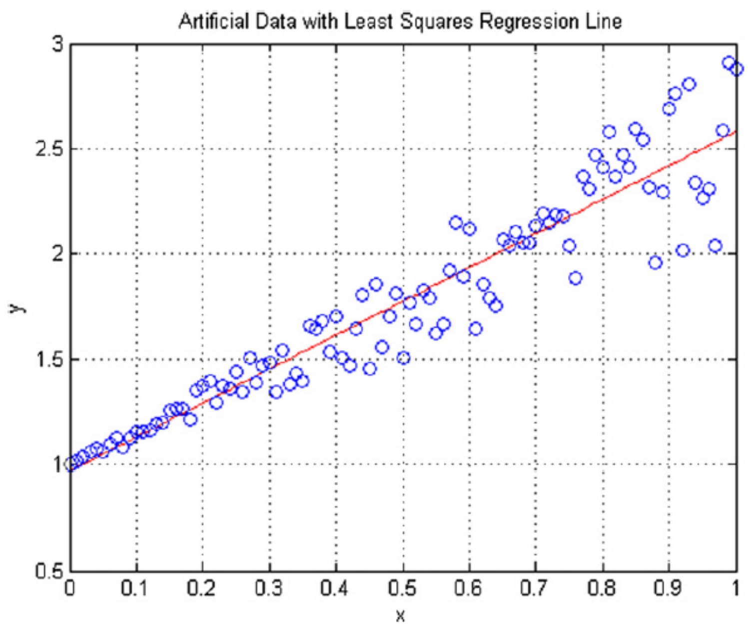
TRY IT! For the matrix `A = [1 2; 3 4; 5 6]` and the vector `y = [4; 1; 2]`, show that `x = inv (A'*A)*A'*y`, `x = pinv (A)*y`, and `x = A\y` all produce the same result for x.

```
>> A = [1 2; 3 4; 5 6];
>> y = [4; 1; 2];

>> x = inv(A'*A)*A'*y
x =
   -4.3333
    3.8333

>> x = pinv(A)*y
x =
   -4.3333
    3.8333

>> x = A\y
x =
   -4.3333
    3.8333
```

TRY IT! Consider the artificial data created by `x = 0:.01:1` and `y = 1 + x + x.*rand (size (x));`. Do a least squares regression with an estimation function defined by

$\hat{y}(x) = \alpha_1 x + \alpha_2$. Plot the data points along with the least squares regression line, as shown in the Figure 13.3. Note that we expect $\alpha_1 = 1$ and $\alpha_2 = 1.5$ based on this data.

```
>> x = [0:.01:1]';
>> y = 1 + x + x.*rand(size(x));
>> A = [x, ones(size(x))];
% the column of ones comes from x^0,
% the basis function associated with alpha(1)
>> alpha = inv(A'*A)*A'*y
alpha =
    1.6111
    0.9686
>> hold on
>> plot(x, alpha(1)*x + alpha(2),'r')
>> plot(x,y,'bo')
>> xlabel('x')
>> ylabel('y')
```



Plotting resulting from execution of previous code. Estimation data and regression curve $\hat{y}(x) = \alpha_1 x + \alpha_2$.

Functions and Operators

```
\        polyfit
pinv     polyval
```

3. Chapter III: Differential Equations

This chapter is a review of ordinary differential equations.

3.1. Simple Differential equations

In this section we present two simple examples to show the importance of differential equations in applications; especially, in Engineering fields.

Example 4.1.1

The current and voltage in a capacitor are related by

$$i_C(t) = C\frac{dv_C}{dt} \tag{4.1}$$

Where $i_C(t)$ is the current through the capacitor, v_C is the voltage across the capacitor, and the constant C is the capacitance in farads (F). For this example, $C = 1\ F$ and the capacitor is being charged by a constant current I. Find the voltage v_C across this capacitor as a function of time given that the voltage at some reference time $t = 0$ is v_0.

Solution

It is given that the current, as a function of time, is constant, that is,

$$i_C(t) = I = constant \tag{4.2}$$

By substituting equation (4.2) in equation (4.1) we get

$$\frac{dv_C}{dt} = I$$

And by separation of the variables,

$$dv_C = Idt \tag{4.3}$$

Integrating both sides of equation (4.4) we obtain

$$v_C(t) = It + K \tag{4.4}$$

Where K represents the constants of integration of both sides.

We can find the value of the constant K by making use of the initial condition, i.e., at $t = 0$, $v_C = v_0$ and equation (4.4) then becomes

$$v_0 = 0 + K \tag{4.5}$$

Or $k = v_0$ and by substitution in equation (4.4) we get

$$v_C(t) = It + v_0 \tag{4.6}$$

This example shows that when a capacitor is charged with a constant current, a linear voltage is produced across the terminals of the capacitor.

Example 4.1.2

Given that,

$$\frac{di_L}{dt} = \cos(t) \tag{4.7}$$

By separating the variables, we obtain

$$di_L = \cos(t)\,dt \tag{4.8}$$

and integrating both sides we obtain

$$i_L(t) = \sin(t) + K \tag{4.9}$$

where represents the constants of integration of both sides.

We find the value of the constant K by making use of the initial condition. For this example, $\omega = 1$ and thus at $\omega t = t = {}^{\pi}/_{2}$, $i_L = 1$. With these values, equation (4.9) becomes

$$1 = \sin\left(\frac{\pi}{2}\right) + K \tag{4.10}$$

Or $k = 0$, and by substitution into equation (4.9)

$$i_L(t) = \sin(t) \tag{4.11}$$

3.2. Classification of Differential equations

Differential equations are classified by:

1. *Type - Ordinary or Partial.*
2. *Order - The highest order derivative which is included in the differential equation.*
3. *Degree - The exponent of the highest power of the highest order derivative after the differential equation has been cleared of any fractions or radicals in the dependent variable and its derivatives*

For example, the differential equation

$$\left(\frac{d^4y}{dx^4}\right)^2 + 5\left(\frac{d^3y}{dx^3}\right)^4 + 6\left(\frac{d^2y}{dx^2}\right)^6 + 3\left(\frac{dy}{dx}\right)^8 + \frac{y^2}{x^3+1} = ye^{-2x}$$

is an ordinary differential equation of order 4 and degree 2

If the dependent variable y is a function of only a single variable x, that is, if $y = f(x)$, the differential equation which relates y and x is said to be an *ordinary differential equation* and it is abbreviated as ODE.
The differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2 = 5\cos(4t)$$

is an ODE with constant coefficients.

The differential equation

$$x^2\frac{d^2y}{dt^2} + x\frac{dy}{dt} + (x^2 - n^2) = 0$$

is an ODE with variable coefficients.

If the dependent variable y is a function of two or more variables such as $y = f(x, t)$, where x and t are independent variables, the differential equation that relates y, x,and t is said to be a *partial differential equation* and it is abbreviated as *PDE*.
An example of a partial differential equation is the well-known *one-dimensional wave equation* shown below.

$$\frac{\partial^2y}{dt^2} = a^2\frac{\partial^2y}{dx^2}$$

Most engineering problems are solved with ordinary differential equations with constant coefficients; however, partial differential equations provide often quick solutions to some practical applications as illustrated with the following three examples, see [1].

Example 4.2.1

The equivalent resistance R_T of three resistors R_1, R_2, and R_3 in parallel is obtained from

$$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

Given that initially $R_1 = 5\ \Omega$, $R_2 = 20\ \Omega$, and $R_3 = 4\ \Omega$, compute the change inR_T if $R_2$is increased by 10% and R_3 is decreased 5% by while R_1 does not change.
Solution
The initial value of the equivalent resistance is $R_T = 5\ \|20\|4 = 2\ \Omega$.

Now, we treat R_2 and R_3 as constants and differentiating R_T with respect to R_1we obtain

$$-\frac{1}{{R_T}^2}\frac{\partial R_T}{\partial R_1} = -\frac{1}{{R_1}^2} \quad \text{or} \quad \frac{\partial R_T}{\partial R_1} = \left(\frac{R_T}{R_1}\right)^2$$

Similarly,

$$\frac{\partial R_T}{\partial R_2} = \left(\frac{R_T}{R_2}\right)^2 \quad \text{and} \quad \frac{\partial R_T}{\partial R_3} = \left(\frac{R_T}{R_3}\right)^2$$

and the total differential dR_T is

$$dR_T = \frac{\partial R_T}{\partial R_1}dR_1 + \frac{\partial R_T}{\partial R_2}dR_2 + \frac{\partial R_T}{\partial R_3}dR_3 = \left(\frac{R_T}{R_1}\right)^2 dR_1 + \left(\frac{R_T}{R_2}\right)^2 dR_2 + \left(\frac{R_T}{R_3}\right)^2 dR_3$$

By substitution of the given numerical values we obtain

$$dR_T = \left(\frac{2}{5}\right)^2 (0) + \left(\frac{2}{20}\right)^2 (2) + \left(\frac{2}{4}\right)^2 (-0.2) = 0.02 - 0.05 = -0.03$$

Therefore, the equivalent resistance decreases by 3%.

Example 4.2.2

In a series RC electric circuit that is excited by a sinusoidal voltage, the magnitude of the impedance Z is computed from $Z = \sqrt{R^2 + X_C^2}$. Initially, $R = 4\ \Omega$ and $X_C = 3\ \Omega$. Find the change in the impedance Z if the resistance R is increased by $0.25\ \Omega$ (6.25%) and the capacitive reactance X_C is decreased by $0.125\ \Omega$ (-4.167%).

Solution

We will first find the partial derivatives $\frac{dZ}{dR}$ and $\frac{dZ}{dX_C}$; then we compute the change in impedance from the total differential dZ. Thus,

$$\frac{\partial Z}{\partial R} = \frac{R}{\sqrt{R^2+X_C^2}} \quad \text{and} \quad \frac{\partial Z}{\partial X_C} = \frac{X_C}{\sqrt{R^2+X_C^2}}$$

And hence,

$$dZ = \frac{\partial Z}{\partial R} dR + \frac{\partial Z}{\partial X_C} dX_C = \frac{R\ dR + X_C\ dX_C}{\sqrt{R^2 + X_C^2}}$$

and by substitution of the given values

$$dR = \frac{4\ (0.25) + 3\ (-0.125)}{\sqrt{4^2 + 3^3}} = \frac{1 - 0.375}{5} = 0.125$$

Therefore, if R increases by 6.25% and X_C decreases by 4.167%, the impedance Z increases by 4.167%.

3.3. Solutions of Ordinary Differential Equations (ODE)

A function $y = f(x)$ is a solution of a differential equation if the latter is satisfied when and its derivatives are replaced throughout by $f(x)$ and its corresponding derivatives. Also, the initial conditions must be satisfied.

For example, a solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ is

$$y = k_1 \sin(x) + k_2 \cos(x)$$

Since y and its second derivative satisfy the given differential equation.

Any linear, time-invariant system can be described by an ODE which has the form

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = \underbrace{b_m \frac{d^m x}{dt^m} + b_{m-1}\frac{d^{m-1}x}{dt^{n-1}} + \ldots + b_1 \frac{dx}{dt} + b_0 x}_{\text{Excitation (Forcing) Function } x(t)} \tag{4.12}$$

NON – HOMOGENEOUS DIFFERENTIAL EQUATION

If the excitation in (4.12) is not zero, that is, if $x(t) \neq 0$, the ODE is called a *non-homogeneous ODE*. If $x(t) = 0$, it reduces to:

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = 0 \tag{4.13}$$

HOMOGENEOUS DIFFERENTIAL EQUATION

The differential equation of (4.13) above is called a *homogeneous ODE* and has different linearly independent solutions denoted as $y_1(t), y_2(t), y_3(t), \ldots, y_n(t)$.

- It's not hard to prove that equation (4.13) has a general solution (see [1]) and presented as

$$y_H(t) = k_1 y_1(t) + k_2 y_2(t) + k_3 y_3(t) + \cdots + k_n y_n(t) \tag{4.14}$$

where the subscript H on the left side is used to emphasize that this is the form of the solution of the homogeneous ODE and $k_1, k_2, k_3, \ldots, k_n$ are arbitrary constants.

In our subsequent discussion, the solution of the homogeneous ODE, i.e., the complementary solution, will be referred to as the *natural response*, and will be denoted as $y_N(t)$ or simply y_N. The particular solution of a non-homogeneous ODE will be referred to as the *forced response*, and will be denoted as $y_F(t)$ or simply y_F.

Accordingly, we express the total solution of the non-homogeneous ODE of (4.12) as:

$$y(t) = \underset{\text{Response}}{y_{\text{Natural}}} + \underset{\text{Response}}{y_{\text{Forced}}} = y_N + y_F \quad (4.15)$$

The natural response y_N contains arbitrary constants and these can be evaluated from the given initial conditions. The forced response y_F, however, contains no arbitrary constants. It is imperative to remember that the arbitrary constants of the natural response must be evaluated from the total response.

3.4. Solution of the Homogeneous ODE

Let the solutions of the homogeneous ODE

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0 \quad (4.16)$$

Will be of the form

$$y = ke^{st} \quad (4.17)$$

Then by substituting equation (4.17) in eq. (1.16) we obtain

$$a_n k s^n e^{st} + a_{n-1} k s^{n-1} e^{st} + \cdots + a_1 k s e^{st} + a_0 k e^{st} = 0$$

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)\, k e^{st} = 0 \quad (4.18)$$

We observe that (4.18) can be satisfied when

$$(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0) = 0 \quad \text{or} \quad k = 0 \quad \text{or} \quad s = -\infty \quad (4.19)$$

but the only meaningful solution is the quantity enclosed in parentheses since the latter two yield trivial (meaningless) solutions. We, therefore, accept the expression inside the parentheses as the only meaningful solution and this is referred to as the *characteristic (auxiliary) equation*, that is,

$$\underbrace{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0 = 0}_{\text{Characteristic Equation}} \quad (4.20)$$

Since the characteristic equation is an algebraic equation of an *nth-power* polynomial, its solutions are $s_1, s_2, s_3, \ldots, s_n$, and thus the solutions of the homogeneous ODE are:

$$y_1 = k_1 e^{s_1 t}, \quad y_2 = k_2 e^{s_2 t}, \quad y_3 = k_3 e^{s_3 t}, \ldots, \quad y_n = k_n e^{s_n t} \quad (4.21)$$

Case 1: Distinct Roots

If the roots of the characteristic equation are *distinct* (different from each another), the solutions of (4.20) are independent and the most general solution is:

$$\underset{\text{FOR DISTINCT ROOTS}}{y_N = k_1 e^{s_1 t} + k_2 e^{s_2 t} + \ldots + k_n e^{s_n t}} \quad (4.22)$$

Case 2: Repeated Roots
If two or more roots of the characteristic equation are *repeated* (same roots), then some of the terms of (4.21) are not independent and therefore (4.22) does not represent the most general solution. If, for example, $s_1 = s_2$, then,

$$k_1e^{s_1t} + k_2e^{s_2t} = k_1e^{s_1t} + k_2e^{s_1t} = (k_1 + k_2)e^{s_1t} = k_3e^{s_1t}$$

and we see that one term of (4.22) is lost. In this case, we express one of the terms of (4.22), say $k_2e^{s_1t}$ as $k_2te^{s_1t}$. These two represent two independent solutions and therefore the most general solution has the form:

$$y_N = (k_1 + k_2)e^{s_1t} + k_3e^{s_3t} + \cdots + k_ne^{s_nt} \qquad (4.23)$$

If there m are equal roots the most general solution has the form:

$$y_N = (k_1 + k_2t + \ldots + k_mt^{m-1})e^{s_1t} + k_{n-i}e^{s_2t} + \ldots + k_ne^{s_nt} \qquad (4.24)$$

FOR M EQUAL ROOTS

Case 3: Complex Roots
If the characteristic equation contains complex roots, these occur as complex conjugate pairs. Thus, if one root is $s_1 = \alpha + \beta i$ where α and β are real numbers, then another root is $s_1 = -\alpha - \beta i$. Then,

$$\begin{aligned} k_1e^{s_1t} + k_2e^{s_2t} &= k_1e^{-\alpha t + j\beta t} + k_2e^{-\alpha t - j\beta t} = e^{-\alpha t}(k_1e^{j\beta t} + k_2e^{-j\beta t}) \\ &= e^{-\alpha t}(k_1\cos\beta t + jk_1\sin\beta t + k_2\cos\beta t - jk_2\sin\beta t) \\ &= e^{-\alpha t}[(k_1 + k_2)\cos\beta t + j(k_1 - k_2)\sin\beta t] \\ &= e^{-\alpha t}(k_3\cos\beta t + k_4\sin\beta t) = e^{-\alpha t}k_5\cos(\beta t + \varphi) \end{aligned} \qquad (4.25)$$

FOR TWO COMPLEX CONJUGATE ROOTS

If (4.25) is to be a real function of time, the constants $k_1$and $k_2$must be complex conjugates. The other constants k_3, k_4, k_5, and the phase angle φ are real constants. The forced response can be found by

- *The Method of Undetermined Coefficients*

3.5. Using the Method of Undetermined Coefficients for the Forced Response

For simplicity, we will only consider ODEs of order 2.
Consider the non-homogeneous ODE

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(x) \qquad (4.26)$$

where a, b, and c are real constants.
We have learned that the total (complete) solution consists of the summation of the natural and forced responses.

For the natural response, if y_1 and y_2 are any two solutions of (4.26), the linear combination, $y_3 = k_1 y_1 + k_2 y_2$ where k_1 and k_2 are arbitrary constants, is also a solution, that is, if we know the two solutions, we can obtain the most general solution by forming the linear combination of y_1 and y_2. To be certain that there exist no other solutions, we examine the Wronskian Determinant defined below.

$$W(y_1, y_2) \equiv \begin{bmatrix} y_1 & y_2 \\ \frac{d}{dx} y_1 & \frac{d}{dx} y_2 \end{bmatrix} = y_1 \frac{d}{dx} y_2 - y_2 \frac{d}{dx} y_1 \neq 0 \quad (4.27)$$

WRONSKIAN DETERMINANT

If (4.27) is true, we can be assured that all solutions of (4.26) are indeed the linear combination of y_1 and y_2.
The forced response is obtained by observation of the right side of the given ODE as it is illustrated by the examples that follow.

Example 4.5.1

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 0 \quad (4.28)$$

subject to the initial conditions $y(0) = 3$ and $y'(0) = 4$ where $y' = dy/dt$.

Solution
This is a homogeneous ODE and its total solution is just the natural response found from the characteristic equation $s^2 + 4s + 3 = 0$ whose roots are $s_1 = -1$ and $s_2 = -3$. The total response is:

$$y(t) = y_N(t) = k_1 e^{-t} + k_2 e^{-3t} \quad (4.29)$$

The constants k_1 and k_2 are evaluated from the given initial conditions. For this example,

$$y(0) = 3 = k_1 e^0 + k_2 e^0 \quad \textit{or}$$

$$k_1 + k_2 = 3 \quad (4.30)$$

Also,

$$y'(0) = 4 = \frac{dy}{dt}\Big|_{t=0} = k_1 e^{-t} - 3k_2 e^{-3t}\Big|_{t=0}$$

$$-k_1 - 3k_2 = 4 \quad (4.31)$$

Simultaneous solution of (4.30) and (4.31) yields $k_1 = 6.5$ and $k_2 = -3.5$. By substitution into (4.29), we obtain

$$y(t) = y_N(t) = 6.5\, e^{-t} - 3.5\, e^{-3t} \quad (4.32)$$

Check with MATLAB:

Try it! In command window

```
>> y= dsolve ('D2y+4*Dy+3*y=0', 'y(0)=3', 'Dy(0)=4')
 y =
    (-7/2*exp(-3*t)*exp(t)+13/2)/exp(t)
>> pretty(y)
         - 7/2 exp (-3 t) exp(t) + 13/2
         ------------------------------------
                  exp(t)
```

The function $y = f(x)$, of relation (4.32), shown in Figure 4.5.1, was plotted with the use of the MATLAB script

Try it! In Editor window

```
1. y=dsolve ('D2y+4*Dy+3*y=0', 'y(0)=3', 'Dy(0)=4');
2. ezplot (y, [0 5])
3. xlabel ('t')
4. title ('13/2 exp(-t)-7/2 exp(-3t)')
```

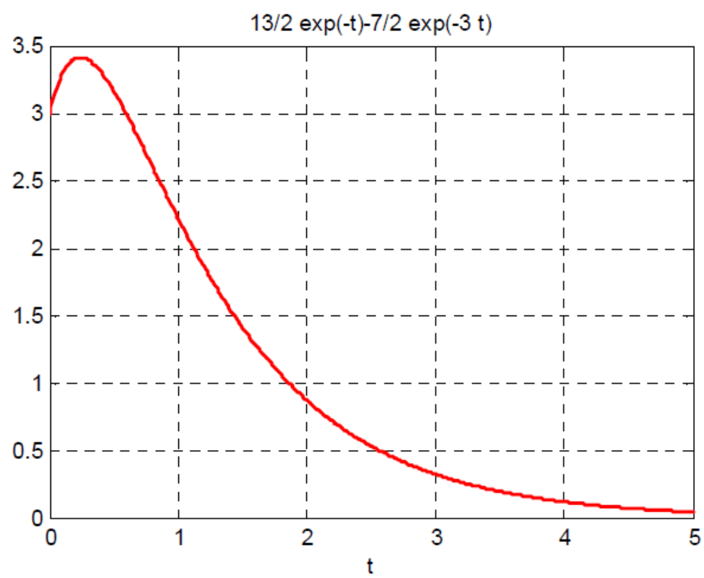


Figure 4.5.1: Plot pf the function $y = f(x)$ for example 4.5.1

Example 4.5.2

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = 3e^{-2t} \qquad (4.33)$$

subject to the initial conditions $y(0) = 1$ and $y'(0) = -1$.

Solution

The left side of (4.33) is the same as that of Example 4.5.1; Therefore,

$$y_N(t) = k_1 e^{-t} + k_2 e^{-3t} \quad (4.34)$$

(We must remember that the constants k_1 and k_2 must be evaluated from the total response). To find the forced response, we assume a solution of the form

$$y_F = Ae^{-2t} \quad (4.35)$$

by substituting (4.35) into the given ODE of (4.33). Then,

$$4Ae^{-2t} - 8Ae^{-2t} + 3Ae^{-2t} = 3e^{-2t} \quad (4.36)$$

from which $A = -3$ and the total solution is

$$y(t) = y_N + y_F = k_1 e^{-t} + k_2 e^{-3t} - 3e^{-2t} \quad (4.37)$$

The constants and are evaluated from the given initial conditions

$$y(0) = k_1 e^{0} + k_2 e^{0} - 3e^{0}$$

Or

$$k_1 + k_2 = 4 \quad (4.38)$$

Also,

$$y'(0) = -1 = \frac{dy}{dt}|_{t=0} = k_1 e^{-t} - 3k_2 e^{-3t}|_{t=0} + 6e^{-2t}|_{t=0}$$

Or

$$-k_1 + 3k_2 = -7 \quad (4.39)$$

Simultaneous solution of (4.38) and (4.39) yields $k_1 = 2.5$ and $k_2 = 1.5$. By substitution into (4.37), we obtain

$$y(t) = y_N + y_F = 2.5\, e^{-t} + 1.5\, e^{-3t} - 3e^{-2t} \quad (4.40)$$

Check with MATLAB:

Try it! In command window

```
>> y=dsolve('D2y+4*Dy+3*y=3*exp(−2*t)', 'y(0)=1', 'Dy(0)=−1')
 y =
    (-3*exp(-2*t)*exp(t)+3/2*exp(-3*t)*exp(t)+5/2)/exp(t)
>> pretty(y)
      -3 exp(-2 t) exp(t) + 3/2 exp(-3 t) exp(t) + 5/2
      -----------------------------------------------------------
                          exp(t)
```

The plot is shown in Figure 5.2 was produced with the MATLAB script

Try it! In Editor window

```
1. y=dsolve('D2y+4*Dy+3*y=3*exp(−2*t)', 'y(0)=1', 'Dy(0)=−1');
2. ezplot (y, [0 8])
3. title ('5/2 exp(-t)-3/2 exp(-3t)-3 exp(-2t)');
4. xlabel('t');
```

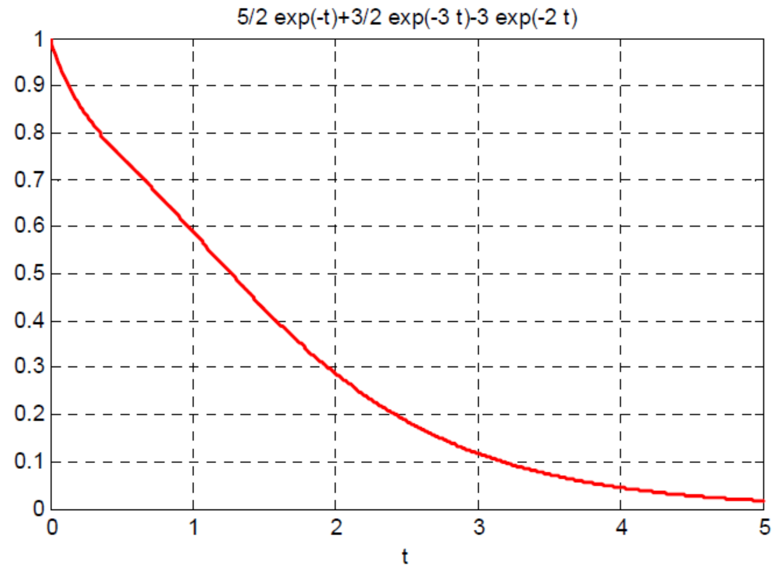


Figure 4.5.2: Plot pf the function $y = f(x)$ for example 4.5.2

Example 4.5.3

Find the total solution of the ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 0 \tag{4.41}$$

subject to the initial conditions $y(0)$ = -1 and $y'(0) = 1$.

Solution

This is a homogeneous ODE and therefore its total solution is just the natural response found from the characteristic equation $s^2 + 6s + 9 = 0$ whose roots are $s_1 = s_2 = -3$ (repeated roots). Thus, the total response is

$$y(t) = y_N(t) = k_1e^{-3t} + k_2te^{-3t} \tag{4.42}$$

Next, we evaluate the constants k_1 and k_2 from the given initial conditions.

$$y(0) = -1 = k_1e^0 + k_2(0)\,e^0$$

Or

$$k_1 = -1 \tag{4.43}$$

Also,

$$y'(0) = 1 = \frac{dy}{dt}|_{t=0} = -3\,k_1e^{-3t} - k_2e^{-3t}|_{t=0} + 3k_2e^{-3t}|_{t=0}$$

Or

$$-3k_1 + k_2 = 1 \tag{4.44}$$

From (4.43) and (4.44) we obtain $k_1 = -1$ and $k_2 = -2$. By substitution into (4.42), we get

$$y(t) = -e^{-3t} - 2te^{-3t} \tag{4.45}$$

Check with MATLAB

Try it! In command window

```
>> y=dsolve ('D2y+6*Dy+9*y=0', 'y(0)=−1', 'Dy(0)=1')
 y =
    -exp(-3*t)-2*exp(-3*t)*t
```

The plot shown in Figure 4.5.3 was produced with the following MATLAB script

Try it! In Editor window

1. y=dsolve ('D2y+6*Dy+9*y=0', 'y(0)=−1', 'Dy(0)=1');
2. ezplot(y,[0 3])
3. title(' ');
4. xlabel(' t');

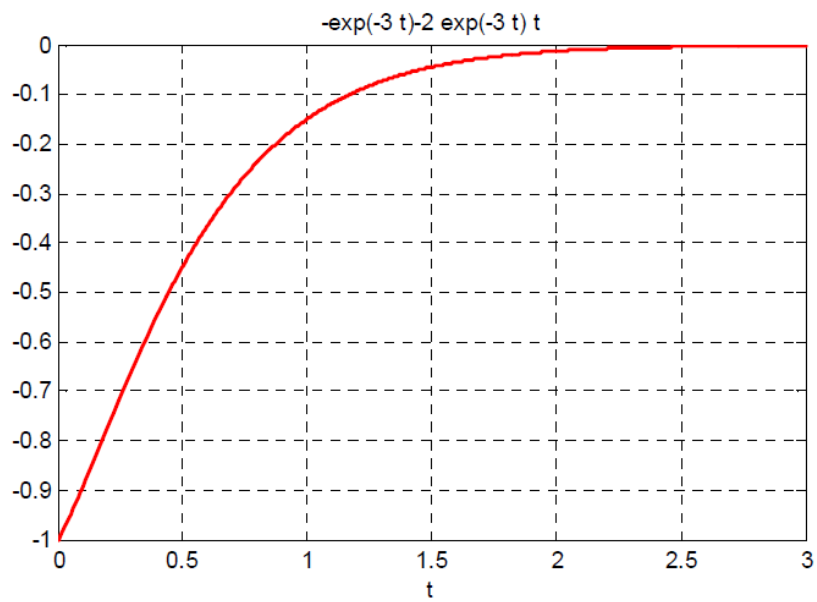


Figure 4.5.3: Plot pf the function $y = f(x)$ for example 4.532

References

[1] Karris, S. T. (2007). *Numerical analysis using MATLAB and Excel*. Orchard Publications.

[2] Siauw, T., & Bayen, A. (2014). *An introduction to MATLAB® programming and numerical methods for engineers*. Academic Press.