**8. Homomorphism, Examples and Basic Concepts**

**Definition(8-1):**Let $\left(G,\*\right),\left(G^{'},∘\right)$ be two groups and $f:\left(G,\*\right)⟶\left(G^{'},∘\right)$ be a mapping, then $f$ is called a homomorphism iff $f\left(a\*b\right)=f(a)∘f\left(b\right)∀a,b\in G$.

**Example(8-2):** Let $f:\left(R,+\right)⟶\left(R^{+},∙\right), \ni f\left(a\right)=2^{a} ∀a\in R$. Is $f$ a homo. ?

**Solution:** let$ a,b\in R⟹f\left(a+b\right)=2^{a+b}=2^{a}∙2^{b}=f(a)∙f(b)$

thus, $f$ is a homo.

**Example(8-3):** Let $f:\left(Z,+\right)⟶\left(Z,+\right), \ni f\left(x\right)=3x+2 ∀x\in Z$. Is $f$ a homo. ?

**Solution:** let$ x,y\in Z⟹f\left(x+y\right)=3\left(x+y\right)+2$

$=3x+3y+2…1$

$$f\left(x\right)+f\left(y\right)=\left(3x+2\right)+\left(3y+2\right)=3x+3y+4…2$$

We have $1\ne 2⟹f\left(x+y\right)\ne f\left(x\right)+f(y)$

Therefore, $f$ is not a homo.

**Example(8-4):** Let $f:\left(S\_{3},∘\right)⟶\left(S\_{3},∘\right), \ni f\left(x\right)=x ∀x\in S\_{3}$. Is $f$ a homo. ? (**Homework**)

**Example(8-5):** Let $f:\left(Z\_{6},+\_{6}\right)⟶\left(Z\_{6},+\_{6}\right), \ni f\left(x\right)=x ∀x\in Z\_{6}$. Is $f$ a homo. ? (**Homework**)

**Example(8-6):** Let $f:\left(R,+\right)⟶\left(Z,+\right), \ni f\left(a\right)=2a-1 ∀a\in R$. Is $f$ a homo. ?

**Solution:** $f\left(a+b\right)=2\left(a+b\right)-1=2a+2b-1…1$

$$f\left(a)+f(b\right)=\left(2a-1\right)+\left(2b-1\right)=2a+2b-2…2$$

We have $1\ne 2⟹f\left(a+b\right)\ne f\left(a\right)+f(b)$

Therefore, $f$ is not a homo.

**Example(8-7):** Let $f:\left(Z,+\right)⟶\left(\left\{1,-1\right\},∙\right),$

$ \ni f\left(a\right)=\left\{\begin{array}{c} 1 a even\\ -1 a odd\end{array}\right.∀a\in Z$. Is $f$ a homo. ?

**Solution:** let $a,b\in Z$

1. $a,b\in E$

$$f\left(a+b\right)=1, \left(a+b\in E\right), f\left(a\right)∙f\left(b\right)=1∙1=1$$

1. $a,b\in O⟹a+b\in E$

$$f\left(a+b\right)=1, \left(a+b\in E\right), f\left(a\right)∙f\left(b\right)=-1∙-1=1$$

1. If $a\in E, b\in O⟹a+b\in O$

$$f\left(a+b\right)=-1, \left(a+b\in O\right), f\left(a\right)∙f\left(b\right)=1∙-1=-1$$

Therefore, $f\left(a+b\right)=f\left(a\right)∙f\left(b\right) ∀a,b\in Z⟹ f$ is a homo.

**Example(8-8):** Let $f:\left(G,\*\right)⟶\left(G,\*\right)\ni f\left(a\right)=x\*a\*x^{-1} ∀a\in G$. Is $f$ a homo. ?

**Solution:** let $a,b\in G\ni f\left(a\*b\right)=x\*\left(a\*b\right)\*x^{-1}…1$

$$f\left(a\right)\*f\left(b\right)=\left(x\*a\*x^{-1}\right)\*\left(x\*b\*x^{-1}\right)$$

$$=x\*\left(a\*b\right)\*x^{-1}…2$$

We have $1=2⟹$ therefore, $f$ is a homo.

**Example(8-9):** Let $f:\left(G,\*\right)⟶\left(G^{'},∙\right) \ni f\left(a\right)=e^{'} ∀a\in G$. Is $f$ a homo. ?

**Solution:** let $a,b\in G\ni f\left(a\*b\right)=e^{'}=e^{'}∙e^{'}=f(a)∙f(b)$

$⟹$ Therefore, $f$ is a trivial homo.

**Example(8-10):** Let $H⊳G$ and $f:\left(G,\*\right)⟶\left(\frac{G}{H},⨂\right) \ni f\left(a\right)=a\*H ∀a\in G$. Is $f$ a homo. ?

**Solution:** let $a,b\in G\ni f\left(a\*b\right)=\left(a\*b\right)\*H…1$

$$f\left(a\right)⨂f\left(b\right)=\left(a\*H\right)⨂\left(b\*H\right)=\left(a\*b\right)\*H…2$$

We have $1=2⟹$ Therefore, $f$ is a natural homo.

**Definition(8-11):** Let $f:\left(G,\*\right)⟶\left(G^{'},∘\right)$ be a mapping, then

1. $f$ is called a monomorphism (mono.) iff $f$ is a homo. and one to one.
2. $f$ is called an epimorphism (epi.) iff $f$ is a homo. and onto.
3. $f$ is called an isomorphism (iso.) iff $f$ is a homo., one to one and onto.

**Definition(8-12):** Any two groups $\left(G,\*\right),\left(G^{'},∘\right)$ are isomorphic iff there is an isomorphism map between them and denoted by $G≅G^{'}$.

This means,$ G≅G^{'}⟺∃f:\left(G,\*\right)⟶\left(G^{'},∘\right)$ and $f$ is an isomorphism.

**Example(8-13):** Let $(G=\left\{2^{n}:n\in Z\right\},∙)$, show that $(Z,+)≅(G,∙)$.

**Solution:** define $f:\left(Z,+\right)⟶\left(G,∙\right)\ni f\left(n\right)=2^{n} ∀n\in Z$

Homo.? let $n\_{1},n\_{2}\in Z⟹f\left(n\_{1}+n\_{2}\right)$

$=2^{n\_{1}+n\_{2}}=2^{n\_{1}}∙2^{n\_{2}}=f(n\_{1})∙f(n\_{2})⟹f$ is a homo.

One to one? let $f\left(n\_{1}\right)=f(n\_{2})$, to prove $n\_{1}=n\_{2}$

$2^{n\_{1}}=2^{n\_{2}}⟹n\_{1}=n\_{2}⟹f$ is a one to one

Onto? $R\_{f}=\left\{f\left(n\right):n\in Z\right\}=\left\{2^{n}:n\in Z\right\}=G⟹f$ is an onto

$⟹f$ is an isomorphism $⟹(Z,+)≅(G,∙)$

**Theorem(8-14):** Let $f:\left(G,\*\right)⟶\left(G^{'},∙\right)$ be an isomorphism, then

1. $f\left(e\right)=e^{'}$ such that $e$ the identity of $G$.

**Proof:** let $a\in G⟹a\*e=a⟹f\left(a\*e\right)=f(a)$

$$f\left(a)∙f(e\right)=f(a)$$

Let $f\left(a\right)\in G^{'}⟹f\left(a\right)∙f\left(e\right)=f(a)∙e^{'}$

$⟹f\left(e\right)=e^{'}$.

1. $f\left(a^{-1}\right)=(f\left(a\right))^{-1}∀a\in G$

**Proof:** let $a\in G⟹a\*a^{-1}=e⟹f\left(a\*a^{-1}\right)=f\left(e\right)=e^{'}$

$$f\left(a\right)∙f\left(a^{-1}\right)=f\left(e\right)=e^{'}$$

let $f\left(a\right)\in G^{'}⟹f\left(a\right)∙\left(f\left(a\right)\right)^{-1}=e^{'}$

$f\left(a\right)∙f\left(a^{-1}\right)=f(a)∙ \left(f\left(a\right)\right)^{-1}⟹\left(a^{-1}\right)=(f\left(a\right))^{-1}$.

1. If $(H,\*)$ is a subgroup of a group $(G,\*)$, then $(f(H),∙)$ is a subgroup of $\left(G^{'},∙\right)$.

**Proof:** $f\left(H\right)=\{f\left(x\right):x\in H\}⊆G^{'}$

$$e\in H⟹f(e)\in f(H)⟹e^{'}\in f(H)\ne ∅$$

Let $a,b\in f(H)$, to prove $a∙b^{-1}\in f(H)$

$$a\in f\left(H\right)⟹a=f(x)\ni x\in H$$

$$b\in f\left(H\right)⟹b=f(y)\ni y\in H$$

$$⟹x\*y^{-1}\in H⟹a∙b^{-1}=f\left(x\right)∙\left(f\left(y\right)\right)^{-1}=f(x)∙f(y^{-1})$$

$$=f\left(x\*y^{-1}\right)⟹a∙b^{-1}=f(x\*y^{-1})\in f(H)$$

1. If $(K,∙)$ is a subgroup of $\left(G^{'},∙\right)$, then $(f^{-1}(K),\*)$ is a subgroup of $(G,\*)$.

**Proof:** $f^{-1}\left(K\right)=\{x\in G:f(x)\in K\}⊆G$

$$f\left(e\right)=e^{'}⟹e\in f^{-1}\left(K\right)⟹f^{-1}\left(K\right)\ne ∅$$

Let $x,y\in f^{-1}\left(K\right)$, to prove $x\*y^{-1}\in f^{-1}\left(K\right)$

$$x\in f^{-1}\left(K\right)⟹f(x)\in K$$

$$y\in f^{-1}\left(K\right)⟹f(y)\in K$$

$$f(x)∙\left(f\left(y\right)\right)^{-1}\in K⟹f(x)∙f(y^{-1})\in K⟹f(x\*y^{-1})\in K$$

$⟹x\*y^{-1}\in f^{-1}\left(K\right)⟹(f^{-1}(K),\*)$ is a subgroup of $(G,\*)$.

1. If $H⊳G$ and $f$ an onto, then $f(H)⊳G^{'}$.

**Proof:** let $y\in G^{'}, a\in f(H)$, to prove $y∙a∙y^{-1}\in f(H)$

$y\in G^{'}$ and $f$ is an onto $⟹∃x\in G\ni f\left(x\right)=y$

$$a\in f\left(H\right)⟹a=f(h)\ni h\in H$$

$x\in G,h\in H$ and $H⊳G⟹x\*h\*x^{-1}\in H$

$$⟹f(x\*h\*x^{-1})\in f(H)⟹f(x)∙f(h)∙f(x^{-1})\in f(H)$$

$⟹y∙a∙y^{-1}\in f(H)⟹f(H)⊳G^{'}$.

1. If $K⊳G^{'}$, then $f^{-1}(K)⊳G$.

**Proof:** $(f^{-1}\left(K\right),\*)$ is a subgroup of $(G,\*)$, to prove $f^{-1}(K)⊳G$

Let $x\in G⟹f\left(x\right)=y\in G^{'}$

$$a\in f^{-1}(K)⟹f(a)\in K$$

$f\left(x\right)\in G^{'}, f(a)\in K$ and $K⊳G^{'}$

$$f\left(x\right)∙f(a)∙\left(f\left(x\right)\right)^{-1}\in K⟹f\left(x\right)∙f(a)∙f(x^{-1})\in K$$

$⟹f(x\*a\*x^{-1})\in K⟹x\*a\*x^{-1}\in f^{-1}(K)⟹f^{-1}(K)⊳G$.

**Theorem(8-15):** The relation of isomorphic is an equivalent.

**Proof:** Reflexive: to prove$ \left(G,\*\right)≅\left(G,\*\right), ∃ i:\left(G,\*\right)⟶\left(G,\*\right)\ni i\left(x\right)=x ∀x\in G$ and $i$ is a homomorphism, one to one and onto, thus $i$ is an isomorphism$ ⟹\left(G,\*\right)≅\left(G,\*\right)$.

Symmetric: let $\left(G,\*\right)≅\left(G^{'},∙\right)$, to prove $\left(G^{'},∙\right)≅\left(G,\*\right)$, $∃ f:\left(G,\*\right)⟶\left(G^{'},∙\right)\ni f$ is an isomorphism, $f$ is a bijective

$⟹∃f^{-1}:\left(G^{'},∙\right)⟶(G,\*)⟹f^{-1}$ is an one to one and onto, to prove $f^{-1}$ is a homomorphism, let $a,b\in G^{'}, f $is an onto$⟹∃x,y\in G\ni f\left(x\right)=a, f\left(y\right)=b, f^{-1}\left(a∙b\right)=f^{-1}\left(f\left(x\right)∙f\left(y\right)\right)=f^{-1}\left(f\left(x\*y\right)\right)=x\*y=f^{-1}\left(a\right)\*f^{-1}(b)$

Thus, $f^{-1}$ is a homomorphism, $f^{-1}$ is an isomorphism,

 $⟹\left(G^{'},∙\right)≅\left(G,\*\right)$.

Transitive: let $\left(G,\*\right)≅\left(G^{'},∙\right)$ and $\left(G^{'},∙\right)≅(G^{''},⨀)$, to prove

$\left(G,\*\right)≅(G^{''},⨀), ∃ f:\left(G,\*\right)⟶\left(G^{'},∙\right)\ni f$ is an isomorphism, $∃ g:\left(G^{'},∙\right)⟶(G^{''},⨀)\ni g$ is an isomorphism.$ ∃g∘f:\left(G,\*\right)⟶(G^{''},⨀)\ni g∘f$ is a bijective. Let $a,b\in G, \left(g∘f\right)\left(a\*b\right)=g\left(f\left(a\*b\right)\right)=g\left(f\left(a\right)∙f\left(b\right)\right)=g\left(f\left(a\right)\right)⨀g\left(f\left(b\right)\right)=(g∘f)(a)⨀(g∘f)(b)$

Hence, $g∘f$ is a homomorphism$⟹g∘f$ is an isomorphism

$⟹\left(G,\*\right)≅(G^{''},⨀)⟹≅$ is an equivalent relation.

**Theorem(8-16):** Prove that

1. Every two finite cyclic groups of the same order are isomorphic.

**Proof:** let $\left(G,\*\right), \left(G^{'},∙\right)$ are two finite cyclic groups, $\ni O\left(G\right)=O\left(G^{'}\right)=n$

$G$ is a cyclic $⟹∃a\in G\ni G=\left〈a\right〉=\left\{a^{k},k\in Z\right\}=\{a^{1},a^{2},…,a^{k}=e\}$

$G^{'}$ is a cyclic $⟹∃b\in G^{'}\ni G^{'}=\left〈b\right〉=\left\{b^{n},n\in Z\right\}=\{b,b^{2},…,b^{n}=e\}$

Define $f:\left(G,\*\right)⟶\left(G^{'},∙\right)\ni f\left(a^{k}\right)=b^{k}∀k\in Z, $let $a^{r}=a^{s}⟹r≡s($mod$ n)⟹r-s=ng\ni g\in Z⟹r=ng+s⟹b^{r}=b^{ng+s}=\left(b^{n}\right)^{g}∙b^{s}⟹b^{r}=b^{s}$, thus $f$ is a map.

Let $f(a^{r})=f(a^{s})⟹b^{r}=b^{s}⟹r≡s($mod$ n)⟹r-s=ng⟹r=ng+s⟹a^{r}=\left(a^{n}\right)^{g}∙a^{s}⟹a^{r}=a^{s}⟹f$ is a one to one.

$R\_{f}=\left\{f\left(a^{k}\right):∀k\in Z\right\}=\left\{b^{k}:∀k\in Z\right\}=G^{'}⟹f$ is an onto.

$f\left(a^{r}\*a^{s}\right)=f\left(a^{r+s}\right)=b^{r+s}=b^{r}∙b^{s}=f(a^{r})∙f(a^{s}) ⟹f$ is a homomorphism$⟹f$ is an isomorphism$⟹G≅G^{'}$.

1. Every finite cyclic group is an isomorphism to$(Z\_{n},+\_{n})$.

**Proof:** let $(G,\*)$ be a finite cyclic group $\ni O\left(G\right)=m$

$$G=\left〈a\right〉=\{a^{1},a^{2},…,a^{m}=e\}$$

1. if $m<n⟹O(G)<O(Z\_{n})⟹f$ is not an onto$⟹G≇Z\_{n}$
2. if $m=n⟹G≅Z\_{n}$

define $f:\left(G,\*\right)⟶\left(Z\_{n},+\_{n}\right)\ni f\left(a^{k}\right)=k ∀k\in Z^{+}, $let $a^{r}=a^{s}⟹r≡s($mod$ n)⟹r=s⟹f(a^{r})=f(a^{s})⟹f$ is a map.

Let $f(a^{r})=f(a^{s})⟹r≡s($mod$ n)⟹r=ng+s⟹a^{r}=a^{s}⟹f$ is an one to one.

$f\left(a^{r}\*a^{s}\right)=f\left(a^{r+s}\right)=r+s=r+\_{n}s=f\left(a^{r}\right)+\_{n}f(a^{s})⟹f$ is a homomorphism.

$R\_{f}=\left\{f\left(a^{k}\right):∀k\in Z^{+}\right\}=\left\{k:∀k\in Z^{+}\right\}=Z\_{n}⟹f$ is an onto$⟹f$ is an isomorphism$⟹(G,\*)≅\left(Z\_{n},+\_{n}\right)$.

1. Every two infinite cyclic group are isomorphic.

**Proof:** let $\left(G,\*\right), (G^{'},∙)$ are infinite cyclic groups.

$$G=\left〈a\right〉=\{…,a^{-2},a^{-1},a^{0},a^{1},a^{2},…\}$$

$$G^{'}=\left〈b\right〉=\{…,b^{-2},b^{-1},b^{0},b^{1},b^{2},…\}$$

Define $f:\left(G,\*\right)⟶\left(G^{'},∙\right)\ni f\left(a^{k}\right)=b^{k}∀k\in Z$

* $f$ is a map (**Homework**)
* $f$ is an one to one (**Homework**)
* $f$ is an onto (**Homework**)
* $f$ is a homomorphism (**Homework**)
1. Every infinite cyclic group is an isomorphic to $(Z,+)$.

**Proof:** since $G$ is a cyclic $⟹G=\left〈a\right〉=\{…,a^{-2},a^{-1},a^{0},a^{1},a^{2},…\}$

$$G⟶…,a^{-2},a^{-1},a^{0},a^{1},a^{2},…$$

$$Z⟶…,a^{-2},a^{-1},a^{0},a^{1},a^{2},…$$

Define $f:\left(G,\*\right)⟶(Z,+)\ni f\left(a^{k}\right)=k∀k\in Z$ (**check**)

**Definition(8-17):** Let $\left(G,\*\right)$ be a group, define

1. Hom$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is a homomorphism$\}$
2. Aut$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is an isomorphism$\}$

**Theorem(8-18):** Let $\left(G,\*\right)$ be a group, then

1. $($Aut$\left(G\right),∘)$ is a group.

**Proof:** 1,2 and 3 (**check**)

Inverse: let $f:\left(G,\*\right)⟶\left(G,\*\right)$, $f$ is an isomorphism, since $f$ is a bijective$ ⟹∃f^{-1}:\left(G,\*\right)⟶\left(G,\*\right)$ and since $f$ is an isomorphism $⟹f^{-1}$ is an isomorphism $⟹f^{-1}\in Aut\left(G\right)$ and $f∘f^{-1}=f^{-1}∘f=i⟹($Aut$\left(G\right),∘)$ is a group.

1. $($Aut$\left(G\right),∘)$ is a subgroup of $($Symm$\left(G\right),∘)$.

**Proof:** Aut$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is an isomorphism$\}$

Symm$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is a bijective$\}$

Aut$\left(G\right)\ne ∅$, since $∃i:\left(G,\*\right)⟶\left(G,\*\right)\ni i$ is an isomorphism

Aut$\left(G\right)⊆$ Symm$\left(G\right)$ and $($Aut$\left(G\right),∘)$ is a group

$⟹($Aut$\left(G\right),∘)$ is a subgroup of $($Symm$\left(G\right),∘)$.

**Definition(8-19):** Let $\left(G,\*\right)$ be a group and $x\in G$. Define $f\_{x}:\left(G,\*\right)⟶\left(G,\*\right)\ni f\_{x}\left(a\right)=x\*a\*x^{-1}, ∀a\in G$, then $f\_{x}$ is called an inner automorphism of $G$ and Inn$\left(G\right)=\{f\_{x}:∀x\in G\}$ or I$\left(G\right)=\{f\_{x}:∀x\in G\}$.

**Theorem(8-20):** Let $\left(G,\*\right)$ be a group and $x\in G$, then:

1. $f\_{x}$ is an isomorphism map.

**Proof:** $f\_{x}\left(a\right)\*f\_{x}\left(b\right)=\left(x\*a\*x^{-1}\right)\*(x\*b\*x^{-1})$

$$=x\*a\*\left(x^{-1}\*x\right)\*b\*x^{-1}=x\*a\*b\*x^{-1}=f\_{x}(a\*b)$$

Thus, $f\_{x}$ is a homomorphism.

Let $f\_{x}\left(a\right)=f\_{x}\left(b\right)⟹x\*a\*x^{-1}=x\*b\*x^{-1}⟹a=b⟹f\_{x}$ is an one to one.

$R\_{f\_{x}}=\left\{f\_{x}\left(a\right):∀a\in G\right\}=G⟹f\_{x}$ is an isomorphism map.

1. $( I\left(G\right),∘)$ is a subgroup of $($Aut$\left(G\right),∘)$.

**Proof:** I$\left(G\right)=\{f\_{x}:f\_{x}:\left(G,\*\right)⟶\left(G,\*\right)\ni f\_{x} $is an isomorphism$\}$

Aut$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is an isomorphism$\}$

$$a\in G⟹f\_{e}\in I\left(G\right)\ne ∅$$

$$f\_{e}\left(a\right)=e\*a\*e^{-1}=a⟹ I\left(G\right)⊆Aut\left(G\right)$$

Closure: let $f\_{x},f\_{y} \in I\left(G\right),(f\_{x}∘f\_{y})(a)=f\_{x}(f\_{y}(a))=f\_{x}(y\*a\*y^{-1})=x\*(y\*a\*y^{-1})\*x^{-1}=(x\*y)\*a\*\left(x\*y\right)^{-1}=f\_{x\*y}(a) $

Inverse: let $f\_{x}\in I\left(G\right), x^{-1}\in G⟹f\_{x^{-1}}\in I\left(G\right)$, $f\_{x}∘f\_{x^{-1}}=f\_{x\*x^{-1}}=f\_{e}⟹f\_{x^{-1}}∘f\_{x}=f\_{x^{-1}\*x}=f\_{e}⟹\left(f\_{x}\right)^{-1}=f\_{x^{-1}}⟹( I\left(G\right),∘)$ is a subgroup of $($Aut$\left(G\right),∘)$.

1. $I\left(G\right)⊳Aut\left(G\right)$

**Proof:** I$\left(G\right)=\{f\_{x}:f\_{x}:\left(G,\*\right)⟶\left(G,\*\right)\ni f\_{x} $is an isomorphism$\}$

Aut$\left(G\right)=\{f:f:\left(G,\*\right)⟶\left(G,\*\right)\ni f $is an isomorphism$\}$

Let $g\in Aut\left(G\right), f\_{x}\in I\left(G\right), \left(g∘f\_{x}∘g^{-1}\right)\left(a\right)=g∘f\_{x}\left(g^{-1}\left(a\right)\right)=g(f\_{x}\left(g^{-1}\left(a\right)\right)=g\left(x\*g^{-1}\left(a\right)\*x^{-1}\right)=g\left(x\right)\*a\*g\left(x^{-1}\right)=g\left(x\right)\*a\*\left(g\left(x\right)\right)^{-1}=f\_{g(x)}(a)\in I\left(G\right)⟹I\left(G\right)⊳Aut\left(G\right)$.

**Definition(8-21):** Let $f:\left(G,\*\right)⟶\left(G^{'},∙\right)$ be a group homomorphism, then the kernel of $f$ denoted by ker$f$ and defined by ker$f=\{x\in G:f\left(x\right)=e^{'}\}$

**Example(8-22):** let $f:\left(R,+\right)⟶\left(R^{+},∙\right)\ni f\left(x\right)=3^{x}$, find ker$f ∀x\in R$.

**Solution:** $f$ is a homomorphism (**check**)$ ⟹$ ker$f$ an exist,

ker$f=\left\{x\in R:f\left(x\right)=1\right\}=\left\{x\in R:3^{x}=1\right\}=\{x=0\}$

**Example(8-23):** Let $f:\left(G,\*\right)⟶\left(G^{'},∙\right)\ni f$ is a trivial homomorphism, find ker$f ∀x\in G$.

 **Solution:**$f\left(x\right)=e^{'} ∀x\in G, f $is a homomorphism $⟹$ ker$f $is an exist.

ker$f=\left\{x\in G:f\left(x\right)=e^{'}\right\}=G$.

**Example(8-24):**let $f:\left(Z,+\right)⟶\left(Z\_{3},+\_{3}\right)\ni f\left(x\right)=\left[x\right] ∀x\in Z$, find ker$f ∀x\in Z$.

**Solution:**$f$ is a homomorphism (**check**)

Ker$f=\left\{x\in Z:f\left(x\right)=\left[0\right]\right\}=\left\{x\in Z:\left[x\right]=\left[0\right]\right\}=\{x\in Z:x≡0$(mod 3)$\}=\left\{x\in Z:x=3k ∀k\in Z\right\}=\{0,\pm 3,\pm 6,…\}⊆Z$.

**Theorem(8-25):** Let $f:\left(G,\*\right)⟶\left(G^{'},∙\right)$ be a group homomorphism, then:

1. $($ Ker$f,\*)$ is a subgroup of $\left(G,\*\right)$.

**Proof:** ker$f=\left\{x\in G:f\left(x\right)=e^{'}\right\}⊆G, f\left(e\right)=e^{'}⟹e\in $ker$f\ne ∅$.

Let $a,b\in $ker$f$, $f\left(a\*b^{-1}\right)=f\left(a\right)∙f\left(b^{-1}\right)=f\left(a\right)∙f(b))^{-1}=e^{'}∙\left(e^{'}\right)^{-1}=e^{'}⟹f\left(a\*b^{-1}\right)=e^{'}⟹a\*b^{-1}\in $ker$f⟹($ Ker$f,\*)$ is a subgroup of $\left(G,\*\right)$.

1. Ker$f⊳G$

**Proof:** $($Ker$f,\*)$ is a subgroup of $\left(G,\*\right)$.

Let $x\in G, a\in $Ker$f, f\left(x\*a\*x^{-1}\right)=f\left(x\right)∙f\left(a\right)∙f\left(x^{-1}\right)=f\left(x\right)∙ e^{'}∙\left(f\left(x\right)\right)^{-1}=e^{'}⟹x\*a\*x^{-1}\in $Ker$f⟹$Ker$f⊳G$.

1. Ker$f=\{e\}$ iff $f$ is an one to one.

**Proof:** $(⟹)$ suppose that Ker$f=\{e\}$

Let $f\left(a\right)=f\left(b\right)⟹f\left(a\right)∙\left(f\left(b\right)\right)^{-1}$

$$=f\left(b\right)∙\left(f\left(b\right)\right)^{-1}⟹f\left(a\right)∙f\left(b^{-1}\right)=e^{'}$$

$⟹f\left(a\*b^{-1}\right)=e^{'}⟹a\*b^{-1}\in $Ker$f⟹a\*b^{-1}=e ⟹a=b$

$(⟸)$ let $a\in $Ker$f$

$f\left(a\right)=f\left(e\right)⟹a=e⟹$Ker$f=\{e\}$.