1. **The Jordan-Holder Theorem and Related Concepts.**

**Definition(10-1):**

By a *chain* for a group is meant any finite sequence of subsets of

descending from to with the property that all the pairs are subgroups of .

**Remark(10-2):**

The integer is called the length of the chain. When , then the chain in definition (10-1) will called the trivial.

**Example(10-3):**

Find all chains in a group .

**Solution:** The subgroups of a group are :

The chains of a group are

is a chain of length one

is a chain of length two.

**Example(10-4):**

In the group of integers modulo , the following chains are normal chains:

,

,

,

.

All subgroups are normal, since is a commutative group.

**Definition(10-5):** (***Normal Chain***)

If is a normal subgroup of a group for all , then the chain is called a *normal chain*.

**Example(10-6):**

Find all chains in the following groups and determine their length and type.

* ;
* ;
* (**Homework**);
* (**Homework).**

**Solution:** The subgroups of a group are :

Then the chains in are:

is a trivial chain of length one

is a normal chain of length two

is a normal chain of length two.

The subgroups of a group are :

Then the chains in are:

is a trivial chain of length one

is a normal chain of length two

is a normal chain of length two

is a normal chain of length three.

**Definition(10-7):** (***Composition Chain***)

In the group , the descending sequence of sets

forms a *composition chain* for provided

1. is a subgroup of ,
2. is a normal subgroup of ,
3. The inclusion , where is a normal subgroup of , implies either or .

**Remark(10-8):**

Every composition chain is a normal, but the converse is not true in general, the following example shows that.

**Example(10-9):**

In the group , the normal chain

is not a composition chain, since it may be further refined by inserting of the set or . On other hand,

and

are both composition chains for .

**Example(10-10):**

Find all chains in the following groups and determine their length and type.

* ;
* ;
* (**Homework**).

**Solution:** The subgroups of a group are :

Then the chains in are:

is a trivial chain of length one.

is a normal chain of length two, but it is not composition chain, since there is a normal subgroup in , such that .

is a normal chain of length two, but it is not composition chain, since there is a normal subgroup in , such that .

is a composition chain of length three.

The subgroups of a group are :

Then the chains in are:

is a trivial chain of length one.

is a normal chain of length two.

is a normal chain of length two.

is a normal chain of length two.

is a normal chain of length two.

is a composition chain of length three.

is a composition chain of length three.

**Example(10-11):**

Let be the group of symmetries of the square.

A normal chain for which fails to be a composition chain is

.

**Example(10-12):** (**Homework)**

Determine the following chain whether normal, composition:

.

**Example(10-13):**

The group has no a composition chain, since the normal subgroups of are the cyclic subgroups , a nonnegative integer, Since the inclusion holds for all , there always exists a proper subgroup of any given group.

**Definition(10-14):**

A normal subgroup is called a *maximal normal subgroup* of the group if and there exists no normal subgroup of such that .

**Example(10-15):**

In the group , the cyclic subgroups and are both maximal normal with orders and , respectively.

**Example(10-16):**

Determine the maximal normal subgroups in the group .

**Solution:** The normal subgroups of are:

The maximal normal subgroups of are and , since there is no normal subgroup in containing and .

**Remark(10-17):**

A chain is a composition of a group , if each normal subgroup is a maximal normal subgroup of , for all .

**Example(10-18);**

In the group the chains is a composition of , since

is a maximal normal subgroup of

is a maximal normal subgroup of,

is a maximal normal subgroup of, and

is a composition of , since

is a maximal normal subgroup of

is a maximal normal subgroup of,

is a maximal normal subgroup of.

**Theorem(10-19):**

A normal subgroup of the group is a maximal if and only if the quotient is a simple.

**Proof:**

Let or

Since is a maximal, or is a simple

let be a simple

has two normal subgroups which are and , but

Therefore is a maximal

**Corollary(10-20):**

The group is a simple, if is a prime number.

**Examples(10-21);**

1. Show that is a maximal normal subgroup of .
2. Show that is a maximal normal subgroup of . (**Homework**)

**Solution(1):**

is a prime is a simple (by Corollary (10-20)). From Theorem (10-19), we get that is a maximal normal subgroup of .

**Corollary(10-22):**

A normal chain is a composition of a group , if is a simple group for all.

**Example(10-23);**

Show that is a composition chain of a group .

**Solution:**  is a prime is a simple.

So, we get that is a maximal normal subgroup of .

is a prime is a simple.

So, we get that is a maximal normal subgroup of .

is a prime is a simple.

So, we get that is a maximal normal subgroup of .

is a prime is a simple.

So, we get that is a maximal normal subgroup of .

By corollaries (10-19) and (1-21), we have that is a composition chain of a group .

**Theorem(10-24):**

Every finite group with more than one element has a composition chain.

**Theorem(10-25):** (**Jordan-Holder**)

In a finite group with more than one element, any two composition chains are equivalent.

**Example(10-26):**

In a group , show that the two chains

,

are compositions and equivalent.

**Solution:**

, since ,

, since ,

, since ,

, since .

Therefore, by Jordan-Holder theorem the two chains

,

are compositions and equivalent.

**Exercises(10-27):**

* Check that the following chains represent composition chains for the indicated group.

1. For , the group of integers modulo :

.

1. For , the group of symmetries of the square:

.

1. For , a cyclic group of order:

.

1. For , the symmetric group on symbols:

.

* Find a composition chain for the symmetric group .
* Prove that the cyclic subgroup is a maximal normal subgroup of if and only if is a prime number.
* Establish that the following two composition chains for are equivalent:

,

.

* Find all composition chains for .
* Find all composition chains for .

1. **- Groups and Related Concepts.**

**Definition(11-1):** (**- Group**)

A finite group is said to be *- group* if and only if the order of each element of is a power of fixed prime .

**Definition(11-2):** (**- Group**)

A finite group is said to be *- group* if and only if , where is a prime number.

**Example(11-3):**

Show that is a - group.

**Solution:**  and

is a - group, with

,

,

,

.

**Example(11-4):**

Determine whether is a - group.

**Solution:**  and

is not - group.

**Example(11-5):** (**Homework)**

Determine whether is a - group.

**Examples(11-6):**

* is a - group, since ,
* is a - group, since ,
* is a - group, since .

**Theorem(11-7):**

Let , then is a - group if and only if and are - groups.

**Proof:** Assume that is a - group, to prove that and are - groups.

Since is a - group , for some .

Since group , for some .

So, is a - group.

To prove is a - group.

Let , to prove is a power of .

, ( since is a - group

Suppose that and are - groups, to prove is a - group.

Let , to prove is a power of .

( is a - group)

From (1) and (2), we have and is a - group,

,

Therefore, is a - group

**Examples(11-8):**

Apply theorem(2-7) on .

**Solution:**

is a - group.

By theorem (2-7), and are - groups.

.

or or or or or ,

is a - group is a - group.

is a - group

is a - group

is a - group

is a - group

is a - group .

**Remark(11-9);**

If is a non-trivial - group, then Cent.

**Theorem(11-10):**

Every group of order is an abelian.

**Proof:** Let be a group of order , to prove is an abelian.

Let Cent is a subgroup of .

By Lagrange Theorem ,

or or

If , but this is contradiction with remark(2-9), so .

If

is an abelian.

If

is a cyclic.

Therefore, is an abelian

**Remark(11-11):**

The converse of theorem(2-10) is not true in general, for example is an abelian, but .

**Exercises(11-12):**

* Let and be two normal -subgroups of a finite group . Show that is a normal -subgroup of .
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Determine whether is a -group.
* Show that is a -group.

1. **Sylow Theorems**

**Definition(12-1):** (***Sylow - Subgroup***)

Let be a finite group and is a prime number, a subgroup of a group is called *sylow - subgroup* if

1. is a - group,
2. is not contained in any other - subgroup of for the same prime number .

**Example(12-2);**

Find sylow - subgroups and sylow- subgroup of the group .

**Solution:** The proper subgroups of the group are

1. is not - subgroup.
2. is a - subgroup.
3. is not - subgroup.
4. is a - subgroup.
5. is a - subgroup.
6. is a - subgroup.

**Theorem(12-3):** (**First Sylow Theorem**)

Let be a finite group of order , where is a prime number is not dividing , then has sylow - subgroup of order .

**Example(12-4):**

Find sylow - subgroup of the group .

**Solution:** , and

by first sylow theorem, the group has sylow - subgroup of order .

is a sylow - subgroup.

**Example(12-5):**

Find sylow - subgroup of the group .

**Solution:** , and

by first sylow theorem, the group has sylow - subgroup of order .

is a sylow - subgroup.

**Example(12-6):**

Find sylow - subgroup of the group .

**Solution:** , and

by first sylow theorem, the group has sylow - subgroup of order .

is a sylow - Subgroup.

**Theorem(12-7):**

Let a prime number and be a finite group such that , then has a subgroup of order which is called sylow - subgroup of .

**Example(12-8):**

Are the following groups and have sylow - subgroups.

**Solution:**

, ,

a subgroup such that which is called sylow - subgroup.

Also, a subgroup such that which is called sylow - subgroup.

, is - subgroup.

Every subgroup of is - subgroup, or or or .

**Theorem(12-9):** (**Second Sylow Theorem)**

The number of distinct sylow -subgroups is which is divide the order of .

**Example(12-10):**

Find the distinct sylow -subgroups of .

**Solution:**

,

a subgroup such that .

The number of sylow -subgroups is and

if and

if and

if and

if and

so, there are three sylow -subgroups.

a subgroup such that .

The number of sylow -subgroups is and

if and

if and

if and

So, there is one sylow -subgroup.

**Example(12-11):**

Find the number of sylow -subgroups of such that

**Solution:**

a subgroup such that .

The number of sylow -subgroups is and

if and

if and

if and

if and

So, there are four sylow -subgroups of .

The number of sylow -subgroups is and

if and

if and

if and

if and

So, there are three sylow -subgroups of .

**Remark(12-12):**

The group has exactly one sylow -subgroup if and only if .

**Example(12-13):**

is a sylow -subgroup of ,

So, there is one sylow -subgroup of .

**Exercises(12-14);**

* Show that there is no simple group of order .
* Show that there is no simple group of order .
* Show that there is no simple group of order .
* Show that whether is a sylow.

1. **Solvable Groups and Their Applications**

**Definition(13-1):**

A group is called a solvable group if and only if, there is a finite collection of subgroups of , such that

1. ,
2. ,
3. is a commutative group .

**Theorem(13-2):**

Every commutative group is a solvable group.

**Proof:**

Suppose that is a commutative, to show that is a solvable.

Let and

1. satisfies, since , or ( every subgroup of commutative group is a normal)
2. is a commutative group, or (the quotient of commutative group is a commutative)

So, is a solvable group,

**Example(13-3):**

Show that is a solvable group.

**Solution:** let

1. satisfies, since , is true,
2. To prove is a commutative group

is a commutative group

is a commutative group

Therefore, is a solvable group.

**Example(13-4):** (**Homework)**

Show that is a solvable group.

**Theorem(13-5):**

Every subgroup of a solvable group is a solvable.

**Proof:** let be a subgroup of and is a solvable group.

To prove is a solvable.

Since is a solvable

there is a finite collection of subgroups of , such that

1. ,
2. ,
3. is a commutative group .

Let

Each is a subgroup of .

1. is hold
2. , , since
3. To prove is a commutative group .

Let such that .

To prove is a homomorphism,

?

So, is a homomorphism

is onto ?

is not onto

( by theorem of homomorphism)

so,

and is a commutative

Hence, is a commutative

Therefore, is a commutative

So, is a solvable

**Theorem(13-6):**

Let and is a solvable, then is a solvable.

**Theorem(13-7):**

Let and both are solvable, then is a solvable.

**Proof:** since is a solvable

there is a finite collection of subgroups of , such that

1. ,
2. ,
3. is a commutative group .

Since is a solvable

there is a finite collection of subgroups of , such that

1. ,
2. ,
3. is a commutative group .

To prove is a solvable group.

or

So, there is a finite collection such that

1. .
2. To prove

Let and to prove

1. To prove is a commutative group

is a commutative group and (

is a commutative group

Therefore, is a solvable group

**Exercises(13-8);**

* Show that every -group is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.
* Show that is a solvable group.

1. **Applications of Group Theory**

**14-1 Cayley Theorem**

**Theorem(14-1-1): (Cayley Theorem)**

Every group is an isomorphic to a group of permutations.

This means if is any group, then , where .

**Proof:** define by

To prove is a homomorphism, one to one and onto.

1. is a homomorphism, let

is a homomorphism.

1. is a one to one, let

is a one to one.

1. is a onto,

Therefore,

**Corollary(14-1-2):**

Every finite group of order is an isomorphic to .

**Example(14-1-3):**

Consider the following Cayley table of a group

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

Show that is an isomorphic to a subgroup of .

**Solution:**

,

,

,

,

Hence, is an isomorphic to the subgroup of :

.

**Example(14-1-4): (Homework)**

Let be a group, apply Cayley Theorem on .

**Example(14-1-5): (Homework)**

Show that is an isomorphic to a subgroup of .

**Exercises(14-1-6):**

* Apply Cayley Theorem on .
* Apply Cayley Theorem on .
* Apply Cayley Theorem on.
* Apply Cayley Theorem on.

**14-2 Direct Product**

**Definition(14-2-1):**

Let and be two normal subgroups of , then is called an internal direct product of and ( is a decomposition by and ) if and only if and .

**Example(14-2-2):**

Consider the following Cayley table of a group

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |
|  |  |  |  |  |

*Let*  and , show that is a decomposition by and .

**Solution:**  since is a commutative group

and

Hence, is decomposition by and .

**Example(14-2-3):**

*Let* be any group withand, show that

is a decomposition by and .

**Solution:**

Therefore, is a decomposition by and .

**Example(14-2-4):**

Let be a group. Is has a proper decomposition.

**Solution:** the subgroups of are

Let and

So,

Let and

Therefore, has no proper decomposition.

**Theorem(14-2-5):**

Let and be two subgroups of and , then and .

**Proof:**

Since and

and (by second theorem of isomorphic)

and

and

**Definition(14-2-6):**

Let and be two groups, define such that . Then is a group which is called an external direct product of and .

**Example(14-2-7): (Homework)**

Show that is a group.

**Example(14-2-8):**

Let and . Find .

**Solution:**

o.

**Theorem(14-2-9):**

Let and be two groups, then

1. is an abelian if and only if both and are abelian.
2. .
3. .
4. .
5. .

**Proof:**

1. suppose that is an abelian, to prove and are abelian.

Let

Since is an abelian, then

Hence, is an abelian.

Similarly that is an abelian.

suppose that and are abelian, to prove is an abelian.

Let , to prove

(is an abelian)

(is an abelian)

Therefore, is an abelian.

1. To prove

To prove is a subgroup of

Let

So, is a subgroup of .

To prove

Let and

To prove

Hence, .

1. (**Homework).**
2. To prove .

**Proof:**

Define

is a map ? let and , so is a map

is an one to one ? let , so is a one to one.

is a homomorphism ? , so is a homomorphism

is an onto ? so is an onto.

Therefore,

1. (**Homework)**

**Theorem(14-2-10):**

Let and be two -groups, then is a -group.

**Proof:**

Since is -group

Since is -group

Therefore, is a -group

**Exercises(14-2-11):**

* Let andare subgroups of , show that  is a decomposition.
* Let , show that is a decomposition.
* Find .
* Is an abelian?
* Is an abelian?
* Is an abelian?
* Is an abelian?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?
* Is a -group?