Foundation of Mathematics 2

Chapter 1 Some Types of Functions

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Chapter 2	***System of Numbers***	**Natural Numbers, Construction of Integer Numbers.**
Chapter 3	***Rational Numbers and Groups***	**Construction of Rational Numbers, Binary Operation.**

References

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Chapter One

Some Types of Functions

1. Inverse Function and Its Properties

We start this section by restate some basic and useful concepts.

Definition 1.1.1. (Inverse of a Relation)
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Definition 1.1.2. (Function)

(i) A relation f from A to B is said to be function iff

$$\forall x \in A\ \exists!\, y \in B \text{ such that } (x,y) \in f$$

(ii) A relation f from A to B is said to be function iff

$$\forall x \in A\ \forall y,z \in B, \text{ if } (x,y) \in f \wedge (x,z) \in f, \text{ then } y = z.$$

(iii) A relation f from A to B is said to be function iff

$$(x_1,y_1) \text{ and } (x_2,y_2) \in f \text{ such that if } x_1 = x_2, \text{ then } y_1 = y_2.$$

This property called **the well-defined relation**.

Notation 1.1.3. We write $f(a) = b$ when $(a,b) \in f$ where f is a function; that is, $(a, f(a)) \in f$. We say that b is the **image** of a under f, and a is a **preimage** of b.

Question 1.1.4. From Definition 1.1 and 1.2 that if $f : X \to Y$ is a function, does $f^{-1}: Y \to X$ exist? If Yes, does $f^{-1}: Y \to X$ is a function?

Example 1.1.5.

(i) Let $A = \{1,2,3\}$, $B = \{a,b\}$ and f_1 be a function from A to B defined bellow. $f_1 = \{(1,a),(2,a),(3,b)\}$. Then f_1^{-1} is ------------------------ .

(ii) Let $A = \{1,2,3\}$, $B = \{a,b,c,d\}$ and f_2 be a function from A to B defined bellow. $f_2 = \{(1,a),(2,b),(3,d)\}$. Then f_2^{-1} is ------------------------ .

(iii) Let $A = \{1,2,3\}$, $B = \{a,b,c,d\}$ and f_3 be a function from A to B defined bellow. $f_3 = \{(1,a),(2,b),(3,a)\}$. Then $f_3^{-1}$is ------------------------ .

(iv) Let $A = \{1,2,3\}$, $B = \{a,b,c,\}$ and f_4 be a function from A to B defined bellow. $f_4 = \{(1,a),(2,b),(3,c)\}$. Then f_4^{-1} is ------------------------ .

(v) Let $A = \{1,2,3\}$, $B = \{a,b,c,\}$ and f_5 be a relation from A to B defined bellow. $f_5 = \{(1,a),(1,b),(3,c)\}$. Then f_5 is -------------------- and f_5^{-1} is ----------------- .

Definition 1.1.6. (Inverse Function)

The function $f: X \to Y$ is said to be has inverse if the inverse relation $f^{-1}: Y \to X$ is function.

Example 1.1.7.

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x + 3$, that is,

$$f = \{(x,y) \in \mathbb{R} \times \mathbb{R}: y = x + 3\}$$

$$f = \{(x, f(x)): x \in \mathbb{R}\}$$

$$f = \{(x, x+3) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$f^{-1} = \{(x,y) \in \mathbb{R} \times \mathbb{R}: (y,x) \in f\}$$

$$f^{-1} = \{(x,y) \in \mathbb{R} \times \mathbb{R}: x = y + 3\}$$

$$f^{-1} = \{(x,y) \in \mathbb{R} \times \mathbb{R}: y = x - 3\}$$

$$f^{-1} = \{\big(x, f^{-1}(x)\big): x \in \mathbb{R}\}$$

$$f^{-1} = \{(x, x-3) \in \mathbb{R} \times \mathbb{R}\}.$$

That is $f^{-1}(x) = x - 3$.

f^{-1} is function as shown below.

Let $(y_1, f^{-1}(y_1))$ and $(y_2, f^{-1}(y_2)) \in f^{-1}$ such that $y_1 = y_2$, T. P. $f^{-1}(y_1) = f^{-1}(y_2)$.

Since $y_1 = y_2$, then $y_1 - 3 = y_2 - 3$ (By add -3 to both sides)

$\Longrightarrow f^{-1}(y_1) = f^{-1}(y_2)$.

(ii) $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$; that is,

$$g = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = x^2\}$$

$$g = \{\big(x, g(x)\big): x \in \mathbb{R}\}$$

$$g = \{(x, x^2) \in \mathbb{R} \times \mathbb{R}\}.$$

Then

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (y, x) \in g\}$$

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = y^2\}$$

$$g^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}: y = \pm\sqrt{x}\}$$

$$g^{-1} = \{\big(x, \pm\sqrt{x}\big) \in \mathbb{R} \times \mathbb{R}\}, \text{ that is } g^{-1}(x) = \pm\sqrt{x}.$$

g^{-1} is not function since $g^{-1}(4) = \pm 2$.

Remark 1.1.8: If f is a function, then $f(x)$ is always is an element in the $Ran(f)$ for all x in $Dom(f)$ but $f^{-1}(y)$ may be a subset of $Dom(f)$ for all y in $Cod(f)$.

Definition 1.1.9. Let $f: X \to Y$ be a function and $A \subseteq X$ and $B \subseteq y$.

(i) The set $f(A) = \{f(x) \in Y: x \in A\} = \{y \in Y: \exists x \in A \text{ such that } y = f(x)\}$ is called the **direct image of A by f**.

(ii) The set $f^{-1}(B) = \{x \in X: f(x) \in B\} = \{x \in X: \exists y \in B \text{ such that } f(x) = y\}$ is called the **inverse image of B with respect to f**.

(iii) A function $f: A \to B$ is **one-to-one** (1-1) or **injective** if each element of B appears at most once as the image of an element of A. That is, a function $f: A \to B$ is injective if $\forall x, y \in A,\ f(x) = f(y) \Rightarrow x = y$ **or** $\forall x, y \in A, x \neq y \Rightarrow f(x) \neq f(y)$.

(iv) A function $f: A \to B$ is **onto** or **surjective** if $f(A) = B$, that is, each element of B appears at least once as the image of an element of A. That is, a function $f: A \to B$ is surjective if $\forall y \in B, \exists x \in A$ such that $f(x) = y$.

(v) A function $f : A \to B$ is **bijective** iff it is one-to-one and onto.

Remark 1.1.10: Let $f : X \to Y$ be a function and $A \subseteq X$. If $y \in f(A)$, then $f^{-1}(y) \subseteq A$.

Example 1.1.11.

(i) Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^4 - 1$. $f^{-1}(15) = \{x \in \mathbb{R}: x^4 - 1 = 15\}$

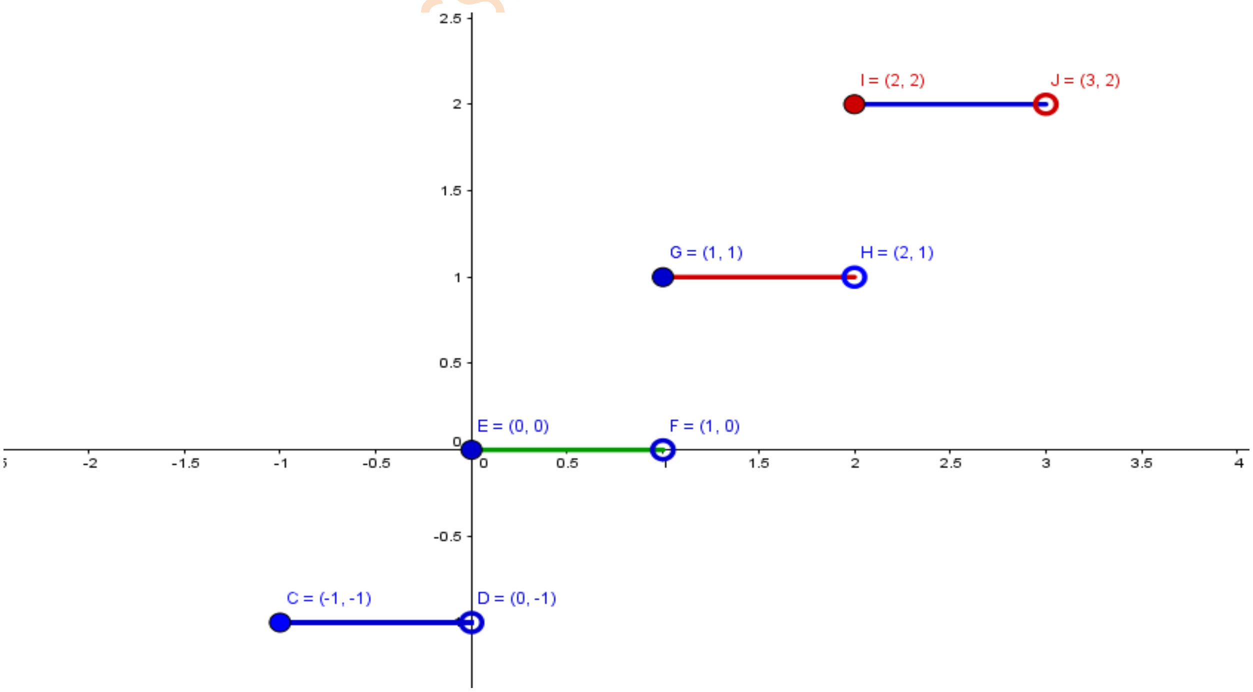
$$= \{x \in \mathbb{R}: x^4 = 16\} = \{-2,2\}.$$

(ii) Let f be a function defined as follows: $f(x) = \begin{cases} -1, & -1 \le x < 0 \\ 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \end{cases}$.

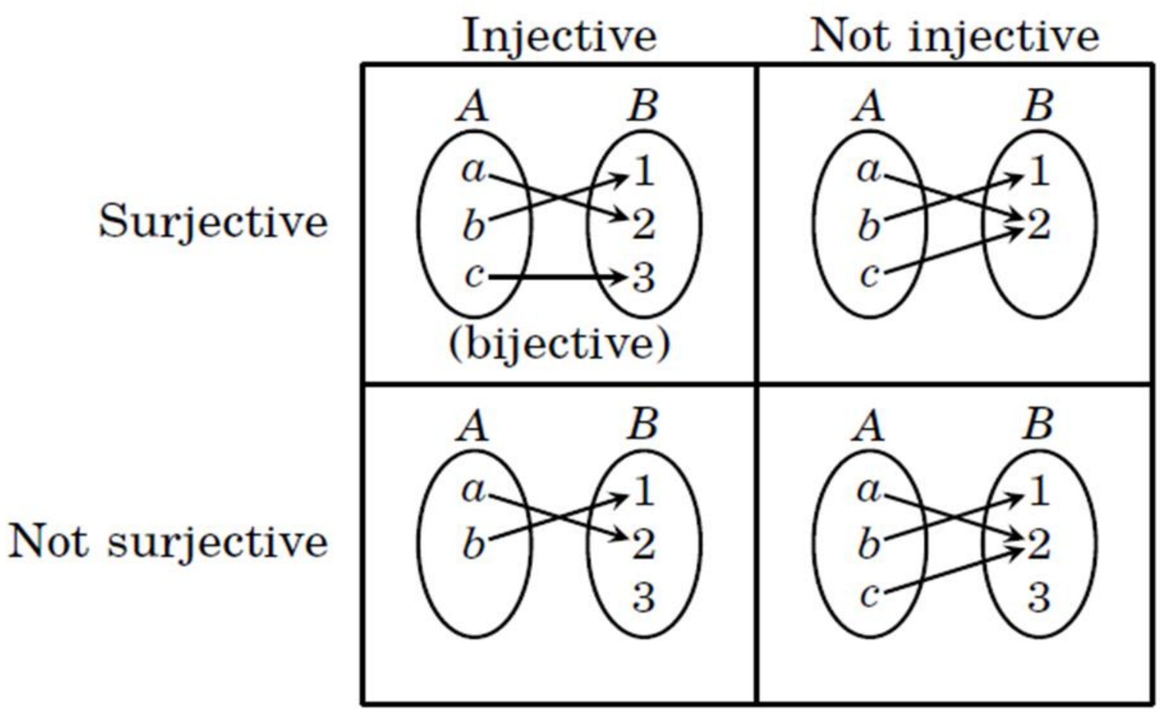
$D(f) = [-1,3), R(f) = \{-1,0,1,2\}$.

$f([-1,-1/2]) = -1$. $f([-1,0]) = \{-1,0\}$.

$f^{-1}(0) = [0,1)$. $f^{-1}([1,3/2]) = [1,2)$.



(iii)



(iv) Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(x) = 3x + 7$.

$f = \{\dots,(-3,-2),(-2,1),(-1,4),(0,7),(1,10),(2,13),\dots\}$.

(a) f is injective. Suppose otherwise; that is,

$f(x) = f(y) \Rightarrow 3x + 7 = 3y + 7 \Rightarrow 3x = 3y \Rightarrow x = y$

(b) f is not surjective. For $b = 2$ there is no a such that $f(a) = b$; that is, $2 = 3a + 7$ holds for $a = -\frac{5}{3}$ which is not in $\mathbb{Z} = D(f)$.

(v) Show that the function $f: \mathbb{R} - \{0\} \to \mathbb{R}$ defined as $f(x) = (1/x) + 1$ is injective but not surjective.

Solution:

We will use the contrapositive approach to show that f is injective.

Suppose $x, y \in \mathbb{R} - \{0\}$ and $f(x) = f(y)$. This means

$\frac{1}{x}+1=\frac{1}{y}+1 \longrightarrow x=y$. Therefore, f is injective.

Function f is not surjective because there exists an element $b=1 \in \mathbb{R}$ for which $f(x)=(1/x)+1 \neq 1$ for every $x \in \mathbb{R}$.

(vi) Show that the function $f: \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \times \mathbb{Z}$ defined by the

formula $f(m,n)= (m+n, m+2n)$, is both injective and surjective.

Solution:

Injective: Let $(m,n),(r,s) \in \mathbb{Z} \times \mathbb{Z} = Dom(f)$ such that $f(m,n)=f(r,s)$. To prove $(m,n)=(r,s)$.

1- $f(m,n)=f(r,s) \Longrightarrow (m+n, m+2n)=(r+s, r+2s)$ Hypothesis

2- $m+n=r+s$ Def. of $\times$

3- $m+2n=r+2s$ Def. of $\times$

4- $m=r+2s-2n$ Inf. (3)

5- $n=s$ and $m=r$ Inf. (2),(4)

6- $(m,n)=(r,s)$ Def. of $\times$

Surjective: Let $(x,y) \in \mathbb{Z} \times \mathbb{Z} = Ran(f)$. To prove $\exists (m,n) \in \mathbb{Z} \times \mathbb{Z} = Dom(f) \ni f(m,n)=(x,y)$.

1- $f(m,n)=(m+n, m+2n)=(x,y)$ Def. of f

2- $m+n=x$ Def. of $\times$

3- $m+2n=y$ Def. of $\times$

4- $m=x-n$ Inf. (2)

5- $n=y-x$ Inf. (3),(4)

6- $m=2x-y$ Inf. (2),(5)

7- $(2x-y, y-x) \in \mathbb{Z} \times \mathbb{Z} = Dom(f)$, $f(2x-y, y-x)=(x,y)$

Theorem 1.1.12. Let $f: A \to B$ be a function. Then f is bijective iff the inverse relation f^{-1} is a function from B to A.

Proof:
Suppose $f: A \to B$ is bijective. To prove f^{-1} is a function from B to A.
$f^{-1} \neq \emptyset$ since f is onto.
$(*)$ Let (y_1, x_1) and $(y_2, x_2) \in f^{-1}$ such that $y_1 = y_2$, to prove $x_1 = x_2$.

(x_1, y_1) and $(x_2, y_2) \in f$ Def. of f^{-1}

(x_1, y_1) and $(x_2, y_1) \in f$ By hypothesis $(*)$

$x_1 = x_2$ Def. of 1-1 on f

$\therefore f^{-1}$ is a function from B to A.

Conversely, suppose f^{-1} is a function from B to A, to prove $f : A \to B$ is bijective, that is, 1-1 and onto.

1-1: Let $a, b \in A$ and $f(a) = f(b)$. To prove $a = b$.

$(a, f(a))$ and $(b, f(b)) \in f$ Hypothesis (f is function)

$(a, f(a))$ and $(b, f(a)) \in f$ Hypothesis ($f(a) = f(b)$)

$(f(a), a)$ and $(f(a), b) \in f^{-1}$ Def. of inverse relation f^{-1}

$a = b$ Since f^{-1} is function

$\therefore f$ is 1-1.

onto: Let $b \in B$. To prove $\exists a \in A$ such that $f(a) = b$.

$(b, f^{-1}(b)) \in f^{-1}$ Hypothesis (f^{-1} is a function from B to A)

$(f^{-1}(b), b) \in f$ Def. of inverse relation f^{-1}

Put $a = f^{-1}(b)$.

$a \in A$ and $f(a) = b$ Hypothesis (f is function)

$\therefore f$ is onto.

Definition 1.1.13.

(i) A function $I_A : A \to A$ defined by $I_A(x) = x$, for every $x \in A$ is called the **identity** function on A. $I_A = \{(x, x): x \in A\}$.

(ii) Let $A \subseteq X$. A function $i_A : A \to X$ defined by $i_A(x) = x$, for every $x \in A$ is called the **inclusion** function on A.

Theorem 1.1.14.

If $f : X \to Y$ is a bijective function, then $f \circ f^{-1} = I_Y$ and $f^{-1} \circ f = I_X$.

Proof: **Exercise.**

Example 1.1.15. Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be a function defined as

$$f(m, n) = (m + n, m + 2n).$$

f is bijective (**Exercise**).

To find the inverse f^{-1} formula, let $f(n, m) = (x, y)$. Then

$(m + n, m + 2n) = (x, y)$. So, the we get the following system

$$\begin{aligned} m + n &= x \dots (1) \\ m + 2n &= y \dots (2) \end{aligned}$$

From (1) we get $m = x - n \quad \dots (3)$

$n = y - x$ Inf (2) and (3) $\dots (4)$

$m = 2x - y$ Rep $(n: y - x)$ or sub(4) in (3)

Define f^{-1} as follows

$f^{-1}(x, y) = (2x - y, y - x)$.

We can check our work by confirming that $f \circ f^{-1} = I_Y$.

$(f \circ f^{-1})(x, y) = f(2x - y, y - x)$

$$= ((2x - y) + (y - x), (2x - y) + 2(y - x))$$

$$= (x, 2x - y + 2y - 2x) = (x, y) = I_Y(x, y)$$

Remark 1.1.16. If $f: X \to Y$ is oneto-one but not onto, then one can still define an inverse function $f^{-1}: Ran(f) \to X$ whose domain in the range of f.

Theorem 1.1.17. Let $f: X \to Y$ be a function.

(i) If $\{Y_j \subseteq Y : j \in J\}$ is a collection of subsets of Y, then

$f^{-1}(\bigcup_{j\in J} Y_j) = \bigcup_{j\in J} f^{-1}(Y_j)$ and $f^{-1}\left(\bigcap_{j\in J} Y_j\right) = \bigcap_{j\in J} f^{-1}(Y_j)$

(ii) If $\{X_i \subseteq X : i \in I\}$ is a collection of subsets of X, then

$f(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} f(X_i)$ and $f(\bigcap_{i\in I} X_i) \subseteq \bigcap_{i\in I} f(X_i)$.

(iii) If A and B are subsets of X such that $A = B$, then $f(A) = f(B)$. The converse is not true.

(iv) If C and D are subsets of Y such that $C = D$, then $f^{-1}(C) = f^{-1}(D)$. The converse is not true.

(v) If A and B are subsets of X, then $f(A) - f(B) \subseteq f(A - B)$. The converse is not true.

(vi) If C and D are subsets of Y, then $f^{-1}(C) - f^{-1}(D) = f^{-1}(C - D)$.

Proof:

(i) Let $x \in f^{-1}(\bigcup_{j\in J} Y_j)$.

$\exists y \in \bigcup_{j\in J} Y_j$ such that $f(x) = y$ Def. of inverse image

$y \in Y_j$ for some $j \in J$ ($f(x) \in Y_j$ for some $j \in J$) Def. of $\cup$

$x \in f^{-1}(Y_j)$ Def. of inverse image

so $x \in \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\cup$

It follow that $f^{-1}(\bigcup_{j\in J} Y_j) \subseteq \bigcup_{j\in J} f^{-1}(Y_j)$ Def. of $\subseteq$ $\ldots\ldots(*)$

Conversely,

If $x \in \bigcup_{j\in J} f^{-1}(Y_j)$, then $x \in f^{-1}(Y_j)$, for some $j \in J$ Def. of $\cup$

So $f(x) \in Y_j$ and $f(x) \in \bigcup_{j \in J} Y_j$ — Def. of inverse and $\cup$

$x \in f^{-1}(\bigcup_{j \in J} Y_j)$ — Def. of inverse f^{-1}

It follow that $\bigcup_{j \in J} f^{-1}(Y_j) \subseteq f^{-1}(\bigcup_{j \in J} Y_j)$ — Def. of $\subseteq$ …..$(**)$

$\therefore f^{-1}(\bigcup_{j \in J} Y_j) = \bigcup_{j \in J} f^{-1}(Y_j)$ — From $(*), (**)$ and Def. of $=$

Example 1.1.18. Let $f: \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function defined as $f(x) = 1$.

$\mathbb{Z}_e \cap \mathbb{Z}_o = \emptyset$. $f(\mathbb{Z}_e \cap \mathbb{Z}_o) = f(\emptyset) = \emptyset$. But $f(\mathbb{Z}_e) \cap f(\mathbb{Z}_o) = \{1\}$.

2. Types of Function

Definitions 1.2.1.

(i) (Constant Function)

The function $f: X \longrightarrow Y$ is said to be **constant function** if there exist a unique element $b \in Y$ such that $f(x) = b$ for all $x \in X$.

(ii) (Restriction Function)

Let $f: X \longrightarrow Y$ be a function and $A \subseteq X$. Then the function $g: A \longrightarrow Y$ defined by $g(x) = f(x)$ all $x \in X$ is said to be **restriction function** of f and denoted by $g = f|_A$.

(iii) (Extension Function)

Let $f: A \longrightarrow B$ be a function and $A \subseteq X$. Then the function $g: X \longrightarrow B$ defined by $g(x) = f(x)$ all $x \in A$ is said to be **extension function** of f from A to X.

(iv) (Absolute Value Function)

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ which defined as follows

$$f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}$$

is called the **absolute value function.**

(v) (Permutation Function)

Every bijection function f on a non empty set A is said to be **permutation** on A.

(vi) (Sequence)

Let A be a non empty set. A function $f: \mathbb{N} \longrightarrow A$ is called a sequence in A and denoted by $\{f_n\}$, where $f_n = f(n)$.

(vii) (Canonical Function)

Let A be a non empty set, R an equivalence relation on A and A/R be the set of all equivalence class. The function $\pi: A \longrightarrow A/R$ defined by $\pi(x) = [x]$ is called the **canonical function**.

(viii) (Projection Function)

Let A_1, A_2 be two sets. The function $P_1: A_1 \times A_2 \longrightarrow A_1$ defined by $P_1(x,y) = x$ for all $(x,y) \in A_1 \times A_2$ is called the **first projection.**

The function $P_2: A_1 \times A_2 \longrightarrow A_2$ defined by $P_2(x,y) = y$ for all $(x,y) \in A_1 \times A_2$ is called the **second projection.**

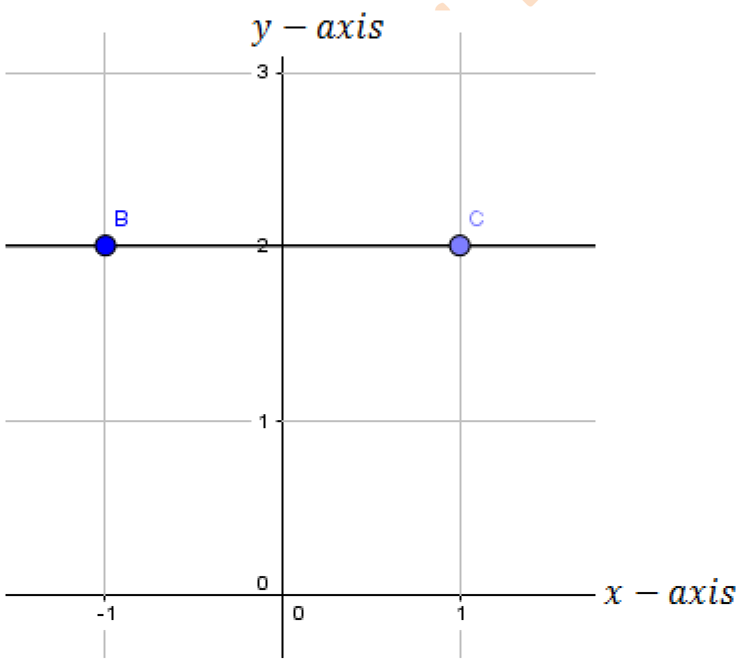
(ix) (Cross Product of Functions)

Let $f: A_1 \longrightarrow A_2$ and $g: B_1 \longrightarrow B_2$ be two functions. The cross product of f with g, $f \times g: A_1 \times B_1 \longrightarrow A_2 \times B_2$ is the function defined as follows:

$$(f \times g)(x,y) = (f(x), g(y)) \text{ for all } (x,y) \in A_1 \times B_1.$$

Examples 1.2.2.

(i)(Constant Function). $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2, \forall x \in \mathbb{R}$. $Dom(f) = \mathbb{R}$, $Ran(f) = \{2\}$, $Cod(f) = \mathbb{R}$.

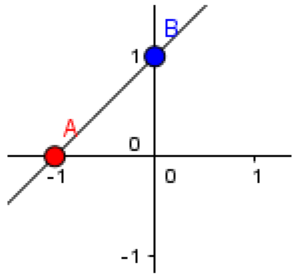


(ii) (Restriction Function). $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x + 1, \forall x \in \mathbb{R}$.

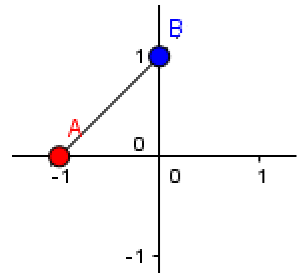
$Dom(f) = \mathbb{R}$, $Ran(f) = \mathbb{R}$, $Cod(f) = \mathbb{R}$. Let $A = [-1,0]$.

$g = f|_A: A \longrightarrow \mathbb{R}$. $g(x) = f(x) = x + 1, \forall x \in A$.

$D(g) = A, R(g) = [0,1], Cod(g) = \mathbb{R}.$



$f(x) = x + 1$



$g = f|_A$

(iii) (Extension Function). $f: [-1,0] \longrightarrow \mathbb{R}, f(x) = x + 1, \forall x \in [-1,0].$

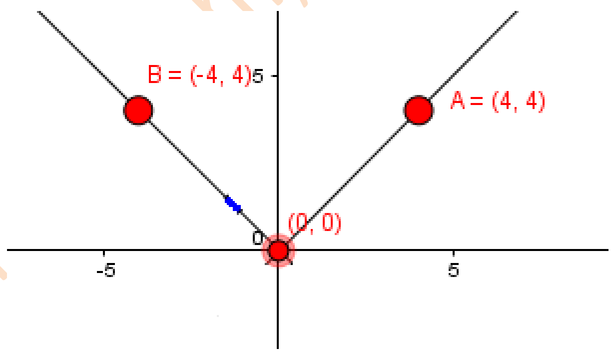
$Dom(f) = [-1,0], R(f) = [0,1], Cod(f) = \mathbb{R}.$

Let $A = \mathbb{R}$. $g: A \longrightarrow \mathbb{R}$. $g(x) = f(x) = x + 1, \forall x \in A.$

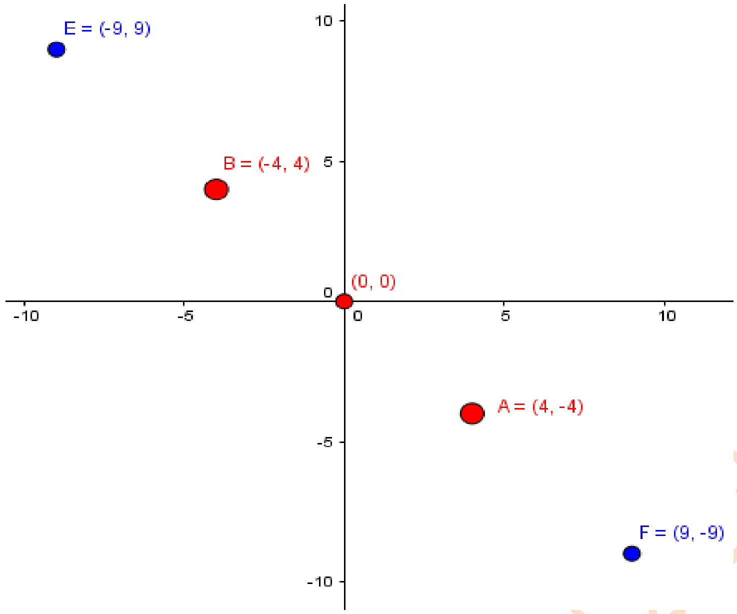
$D(g) = A, \ R(g) = \mathbb{R}, Cod(g) = \mathbb{R}.$

(iv) (Absolute Value Function) $f: \mathbb{R} \longrightarrow \mathbb{R}, \ f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x & x < 0 \end{cases}.$

$Dom(f) = \mathbb{R}, , R(f) = [0, \infty), Cod(f) = \mathbb{R}.$



(v) (Permutation Function). $f: \mathbb{Z} \longrightarrow \mathbb{Z}, \ f(x) = -x, \forall x \in \mathbb{Z}.$ The function is bijective, so it is permutation function. $Dom(f) = \mathbb{Z}, , Ran(f) = \mathbb{Z}, Cod(f) = \mathbb{Z}.$



(vi) (Sequence). $f: \mathbb{N} \longrightarrow \mathbb{Q}, f(n) = \frac{1}{n}, \forall x \in \mathbb{N}$. $\{f_n\} = \{\frac{1}{n}\}_{n=1}^{\infty}$.

(vii) (Canonical Function). Let R be an equivalence relation defined on $\mathbb{Z}$ as follows:

xRy iff $x - y$ is even integer, that is, $R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z}: x - y \text{ even}\}$.

$[0] = \{x \in \mathbb{Z}: x - 0 \text{ even}\} = \{\dots, -4, -2, 0, 2, 4, \dots\} = [2] = [-2] = \cdots$.

$[1] = \{x \in \mathbb{Z}: x - 1 \text{ even}\} = \{\dots, -5, -3, -1, 1, 3, 5, \dots\} = [-1] = [3] = \cdots$.

$\mathbb{Z}/R = \{[0], [1]\}$.

$\pi(0) = [0] = \pi(2) = \pi(-2) = \cdots$.

$\pi(1) = [1] = \pi(-1) = \pi(-3) = \cdots$.

(viii) (Projection Function)

$P_1: \mathbb{Z} \times \mathbb{Q} \longrightarrow \mathbb{Z}, P_1(x, y) = x$ for all $(x, y) \in \mathbb{Z} \times \mathbb{Q}$. $P_1\left(2, \frac{2}{5}\right) = 2$. $P_1(\mathbb{Z}, \frac{2}{5}) = \mathbb{Z}$.

$P_1^{-1}(3) = \{3\} \times \mathbb{Q}$.

(ix) (Cross Product of Functions)

$f: \mathbb{N} \longrightarrow \mathbb{Q}, f(n) = \frac{1}{n}, \forall n \in \mathbb{N}$ and $f: \mathbb{N} \longrightarrow \mathbb{Z}, f(x) = -x, \forall x \in \mathbb{N}$

$f \times g: \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q} \times \mathbb{Z}$, $(f \times g)(x,y) = (f(x), g(y))$

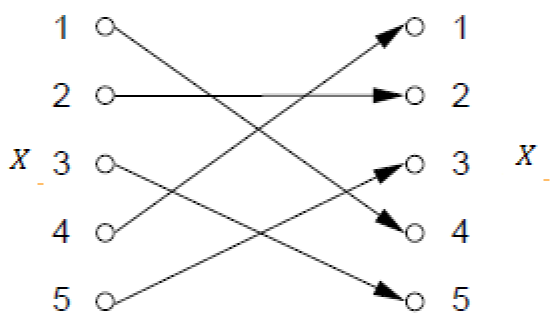
$= (\frac{1}{x}, -y)$ for all $(x,y) \in \mathbb{N} \times \mathbb{N}$.

(x) (Involution Function)

Let X be a finite set and let f be a bijection from X to X (that is, $f: X \longrightarrow X$).
The function f is called an ***involution*** if $\boxed{f = f^{-1}}$. An equivalent way of stating this is

$$\boxed{f(f(x)) = x \quad \text{for all} \quad x \in X.}$$

The figure below is an example of an involution on a set X of five elements. In the diagram of an involution, note that if j is the image of i then i is the image of j.



Exercise 1.2.3.

(i) Let R be an equivalence relation defined on $\mathbb{N}$ as follows:

$$R = \{(x, y) \in \mathbb{N} \times \mathbb{N}: x - y \text{ divisble by } 3\}.$$

1- Find $\mathbb{N}/R$.

2- Find $\pi([0]), \pi([1]), \pi^{-1}([2])$.

(ii) Prove that: the Projection function is onto but not injective.

(iii) Prove that: the Identity function is bijective.

(iv) Prove that: the inclusion function is bijective onto its image.

(v) Let $f: A_1 \longrightarrow A_2$ and $g: B_1 \longrightarrow B_2$ be two functions. If f and g are both 1-1 (onto), then $f \times g$ is 1-1(onto).

(vi) If $f: X \longrightarrow Y$ is a bijective function, then f^{-1} is bijective function.

(vii) If $f: X \longrightarrow Y$ is a bijective function, then

1- $f \circ f^{-1} = I_Y$ is bijective function. **2-** $f^{-1} \circ f = I_X$ is bijective function.

(viii) Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow X$ be functions. If $g \circ f = I_X$, then f is injective and g is onto.

(ix) Let $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function defined as follows:

$$f(x, y) = x^2 + y^2.$$

1- Find the $f(\mathbb{R} \times \mathbb{R})$ (image of f).

2- Find $f^{-1}([0,1])$.

3- Does f 1-1 or onto?

4- Let $A = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x = \sqrt{2 - y^2}\}$. Find $f(A)$.