Foundation of Mathematics 2

Chapter 3 Rational Numbers and Groups

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1. Construction of Rational Numbers

Consider the set

$$V = \{(r,s) \in \mathbb{Z} \times \mathbb{Z} \mid r,s \in Z, s \neq 0\}$$

of pairs of integers. Let us define an equivalence relation on V by putting

$$\boxed{(r,s)\, L^*\, (t,u) \Leftrightarrow ru = st}.$$

This is an equivalence relation. **(Exercise).**

Let

$$[r,s] = \{(x,y) \in V \mid (x,y)\, L^*\, (r,s)\},$$

denote the equivalence class of (r,s) and write $[r,s] = \frac{r}{s}$. Such an equivalence class $[r,s]$ is called a **rational number.**

Example 3.1.1.
(i) $(2,12)\, L^*\, (1,6)$ since $2 \cdot 6 = 12 \cdot 1$,
(ii) $(2,12) \not{L^*}\, (1,7)$ since $2 \cdot 7 \neq 12 \cdot 1$.
(iii) $[0,1] = \{(x,y) \in V | 0y = x1\} = \{(x,y) \in V | 0 = x\} = \{(0,y) \in V | y \in \mathbb{Z}\}$
$= \{(0,\pm 1), (0,\pm 2), \dots\} = [0,y]$.
(iv) $(x,0) \notin V \quad \forall x \in \mathbb{Z}$

Definition 3.1.2. (Rational Numbers)
The set of all equivalence classes $[r,s]$ (rational number) with $(r,s) \in V$ is called the **set of rational numbers** and denoted by $\mathbb{Q}$. The element $[0,1]$ will denoted by 0 and $[1,1]$ by 1.

3.1. 3. Addition and Multiplication on $\mathbb{Q}$
Addition: $\oplus: \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$**;**

$$\boxed{[r,s] \oplus [t,u] = [ru + ts, s\,u]},\ s,u \neq 0.$$

Multiplication: $\odot: \mathbb{Q} \times \mathbb{Q} \longrightarrow \mathbb{Q}$**;**

$$\boxed{[r,s] \odot [t,u] = [rt, su]}\ s,u \neq 0.$$

Remark 3.1.4. The relation $i: \mathbb{Z} \longrightarrow \mathbb{Q}$, defined by $i(n) = [n, 1]$ is 1-1 function, and
$i(n + m) = i(n) \oplus i(m)$,
$i(n \cdot m) = i(n) \odot i(m)$.

Theorem 3.1.5.

(i) $n \oplus m = m \oplus n, \forall\, n, m \in \mathbb{Q}$. (Commutative property of $\oplus$)
(ii) $(n \oplus m) \oplus c = n \oplus (m \oplus c), \forall\, n, m, c \in \mathbb{Q}$. (Associative property of $\oplus$)
(iii) $n \odot m = m \odot n, \forall\, n, m \in \mathbb{Q}$. (Commutative property of $\odot$)
(iv) $(n \odot m) \odot c = n \odot (m \odot c), \forall\, n, m, c \in \mathbb{Q}$. (Associative property of $\odot$)
(v) $(n \oplus m) \odot c = (n \odot c) \oplus (m \odot c)$ (Distributive law of $\odot$ on $\oplus$)
(vi) If $c = [c_1, c_2] \in \mathbb{Q}$ and $c \neq [0,1]$, then $c_1 c_2 \neq 0$.
(vii) (Cancellation Law for $\oplus$).
$m \oplus c = n \oplus c$, for some $c \in \mathbb{Q} \Leftrightarrow m = n$.
(viii) (Cancellation Law for $\odot$).
$m \odot c = n \odot c$, for some $c (\neq 0) \in \mathbb{Q} \Leftrightarrow m = n$.
(ix) $[0,1]$ is the unique element such that $[0,1] \oplus m = m \oplus [0,1] = m, \forall\, m \in \mathbb{Q}$.
(x) $[1,1]$ is the unique element such that $[1,1] \odot m = m \odot [1,1] = m, \forall\, m \in \mathbb{Q}$.

Proof.

(vii) Let $m = [m_1, m_2], n = [n_1, n_2], c = [c_1, c_2] \in \mathbb{Q}, m_i, n_i, c_i \in \mathbb{Z}, i = 1,2$.

$m \oplus c = n \oplus c$	
$\leftrightarrow [m_1, m_2] \oplus [c_1, c_2] = [n_1, n_2] \oplus [c_1, c_2]$	
$\leftrightarrow [m_1 c_2 + c_1 m_2, m_2 c_2] = [n_1 c_2 + c_1 n_2, n_2 c_2]$	Def. of $\oplus$ for $\mathbb{Q}$
$\leftrightarrow (m_1 c_2 + c_1 m_2, m_2 c_2)\, L^* \,(n_1 c_2 + c_1 n_2, n_2 c_2)$	Def. of equiv. class
$\leftrightarrow (m_1 c_2 + c_1 m_2) n_2 c_2 = (n_1 c_2 + c_1 n_2)\, m_2 c_2$	Def. of L^*
$\leftrightarrow ((m_1 n_2) c_2 + (n_2 m_2) c_1) c_2 = ((n_1 m_2) c_2 + (n_2\, m_2) c_1) c_2$	Properties of $+$ and $\cdot$ in $\mathbb{Z}$
$\leftrightarrow (m_1 n_2) c_2 + (n_2 m_2) c_1 = (n_1 m_2) c_2 + (n_2\, m_2) c_1$	Cancel. law for $\cdot$ in $\mathbb{Z}$
$\leftrightarrow (m_1 n_2) c_2 = (n_1 m_2) c_2$	Cancel. law for$+$ in $\mathbb{Z}$
$\leftrightarrow (m_1 n_2) = (n_1 m_2)$	Cancel. law for $\cdot$ in $\mathbb{Z}$
$\leftrightarrow (m_1, m_2) L^* (n_1, n_2)$	Def. of L^*
$\leftrightarrow [m_1, m_2] = [n_1, n_2]$	Def. of equiv. class

(viii) Let $m = [m_1, m_2]$, $n = [n_1, n_2]$, $c = [c_1, c_2] \in \mathbb{Q}$, m_i, n_i, $c_i \in \mathbb{Z}$ and $c \neq [0,1])$ $i = 1,2$.

$m \odot c = n \odot c$	
$\leftrightarrow [m_1, m_2] \odot [c_1, c_2] = [n_1, n_2] \odot [c_1, c_2]$	
$\leftrightarrow [m_1 c_1, m_2 c_2] = [n_1 c_1, n_2 c_2]$	Def. of $\odot$ for $\mathbb{Q}$

$\leftrightarrow (m_1c_1, m_2c_2)L^* (n_1c_1, n_2c_2)$	Def. of equiv. class
$\leftrightarrow (m_1c_1)(n_2c_2) = (n_1c_1)(m_2c_2)$	Def. of L^*
$\leftrightarrow (m_1n_2)(c_1c_2) = (m_2n_1)(c_1c_2)$	Asso. and comm. of + and $\cdot$ in $\mathbb{Z}$
$\leftrightarrow (m_1n_2) = (m_2n_1)$	$c_1c_2 \neq 0$ and Cancel. law for $\cdot$ in $\mathbb{Z}$
$\leftrightarrow (m_1, m_2)\, L^*(n_1, n_2)$	Def. of L^*
$\leftrightarrow [m_1, m_2] = [n_1, n_2]$	Def. of equiv. class

(i),(ii),(iii),(iv)(v),(vi),(ix),(x) Exercise.

Definition 3.1.6.
(i) An element $[n, m] \in \mathbb{Q}$ is said to be **positive element if** $nm > 0$. The set of all positive elements of $\mathbb{Q}$ will denoted by $\mathbb{Q}^+$**.**
(ii) An element $[n, m] \in \mathbb{Q}$ is said to be **negative element if** $nm < 0$. The set of all positive elements of $\mathbb{Q}$ will denoted by $\mathbb{Q}^-$**.**

Remark 3.1.7. Let $[r, s]$ be any rational number. If $s < -1$ or $s = -1$ we can rewrite this number as $[-r, -s]$; that is, $[r, s] = [-r, -s]$.

Definition 3.1.8. Let $[r, s], [t, u] \in \mathbb{Q}$. We say that $[r, s]$ **less than** $[t, u]$ and denoted by
$[r, s] < [t, u] \Leftrightarrow ru < st$,
where $s, u > 1$ or $s, u = 1$.

Example 3.1.9.
$[2, 5], [7, -4] \in \mathbb{Q}$**.**
$[2,5] \in \mathbb{Q}^+$**,** since $2 = [2,0]$, $5 = [5,0]$ in $\mathbb{Z}$ and $2 \cdot 5 = [2 \cdot 5 + 0 \cdot 0, 2 \cdot 0 + 5 \cdot 0]$

$$= [10,0] = +10 > 0.$$

$[-4,7] \in \mathbb{Q}^-$**,** since $7 = [7,0]$, $-4 = [0,4]$ in $\mathbb{Z}$ and

$$7 \cdot (-4) = [7 \cdot 0 + 0 \cdot 4, 7 \cdot 4 + 0 \cdot 0]$$
$$= [0,32] = -32 < 0.$$

$[-4, 7] < [2,5]$, since $-4 \cdot 5 < 2 \cdot 7$.
$[7, -4] < [2,5]$, since $[7, -4] = [-7, -(-4)] = [-7,4]$, and $-7 \cdot 5 < 2 \cdot 4$.