Atmospheric Thermodynamics

Lecture 4. Entropy and potential temperature

Ref.: Andrews, D. G., 2010: An Introduction to Atmospheric Physics, Cambridge University Press.

The First Law of Thermodynamics, applied to a small change to a closed system, such as a mass of air contained in a cylinder with a movable piston at one end (see Figure 1), can be written:

$$dU = dQ + dW \qquad (1)$$

where dU is the increase of internal energy of the system in the process, dQ is the heat supplied to the system and dW is the work done on the system.

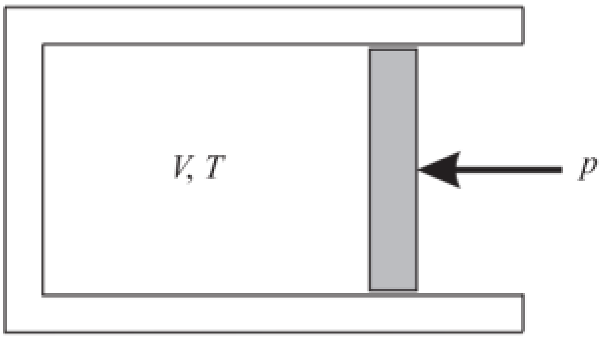

Fig.1 A cylinder of air of V, p and T, closed by a movable piston (shaded).

In terms of functions of state, equation (1) can be written:

$$dU = T\,dS + p\,dV \qquad (2)$$

where S is the entropy of the system. An alternative form of equation (2) is:

$$dH = T\,dS + V\,dp \qquad (3)$$

where $H = U + pV$ is the enthalpy. Since Eq. 2 and Eq. 3 involve functions of state, they apply both for reversible and for irreversible changes. However, we shall mostly restrict our attention to **reversible** changes, for which the equations

$$dQ = T\,dS \qquad (4)$$

$$dW = -pdV \qquad (5)$$

also hold.

For *unit mass* of ideal gas, for which *V* = 1/*ρ*, it can be shown that

$$U = c_v\,T \qquad (6)$$

where c_v is the specific heat capacity at constant volume and is independent of T. Therefore the ideal gas law, $p = R_a T\rho$, implies that, for unit mass of air,

$$H = c_v\,T + R_a T = c_p\,T \qquad (7)$$

where $c_p = c_v + R_a$ is the specific heat capacity of air at constant pressure. On substituting the expression (7) and $V = 1/\rho = R_a T/p$ into equation 3, we get:

$$T\,dS = c_p\,dT - \frac{R_a T}{p}\,dp \qquad (8)$$

Division by *T* gives

$$dS = c_p\frac{dT}{T} - R_a\frac{dp}{p} = c_p\,d(\ln T) - R_a\,d(\ln p) \qquad (9)$$

and integration gives the entropy per unit mass

$$S = c_p \ln T - R_a\,ln\,p + constant = c_p \ln(Tp^{-k}) + S_0 \qquad (10)$$

where $k = R_a/c_p$, which is approximately $\frac{2}{7}$ for a diatomic gas, and S_0 is a constant.

An **adiathermal** process is one in which heat is neither gained nor lost, so that $dQ = 0$. An **adiabatic** process is one that is both adiathermal and reversible; from equations (4 and 5) it follows that $dS = 0$ for such a process. Imagine a cylinder of air, originally at temperature *T* and pressure *p*, that is compressed adiabatically until its pressure equals p_0. We can find its resulting temperature, *θ* say, using equation (9) together with the fact that $dS = 0$ for an adiabatic process, so that,

$$c_p\,d(\ln T) = R_a\,d(\ln p).$$

Integrating and using the end conditions $T = \theta$ and $p = p_0$ then gives

$$c_p \, ln\left(\frac{\theta}{T}\right) = R_a \ln(\frac{p_0}{p})$$

and hence, using $\kappa = R_a/c_p$ again,

$$\theta = T(\frac{p_0}{p})^k. \qquad (11)$$

The quantity θ is called the **potential temperature** of a mass of air at temperature T and pressure p. The value of p_0 is usually taken to be 1000 hPa. Using equation (10) it follows that the potential temperature is related to the specific entropy S by

$$S = c_p \ln \theta + S_1$$

where $S1$ is another constant. By definition, the potential temperature of a mass of air is constant when the mass is subject to an adiabatic change; conversely, the potential temperature will change when the mass is subject to a non-adiabatic (or **diabatic**) change.

As we shall see, the potential temperature is often a very useful concept in atmospheric thermodynamics and dynamics.

Exercises

1. Define: enthalpy , entropy, diabatic change
2. What the difference between adiabatic process and adiathermal process?
3. Starting with the first law of thermodynamics, derive the following relationship of entropy:
$$S = c_p \ln(Tp^{-k}) + S_0$$
4. Starting with the first law of thermodynamics, derive the following relationship between entropy and potential temperature:
$$S = c_p \ln \theta + S_1$$
5. How would potential temperature change if an air mass is subjected to
(1) adiabatic change (2) diabatic change