# Fundamentals of Thermodynamics
## Lecture 1. Math Review

## 1.1 Partial Derivatives and Differentials

- **A differential equation** is an equation which contains one or more terms and the derivatives of one variable (i.e., dependent variable) with respect to the other variable (i.e., independent variable)

$$\frac{dy}{dx} = f(x) \qquad (1)$$

Here "x" is an independent variable and "y" is a dependent variable, e.g. $\frac{dy}{dx} = 5x$

A differential equation contains derivatives which are either partial derivatives or ordinary derivatives. The derivative represents a rate of change, and the differential equation describes a relationship between the quantity that is continuously varying with respect to the change in another quantity.

- **Order of Differential Equation**

The order of the differential equation is the order of the highest order derivative present in the equation. Here some examples for different orders of the differential equation are given:

- $\frac{dy}{dx} = 3x + 2\,, \quad \textit{The order of the equation is } 1$
- $\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right) + y = 0. \quad \textit{The order is } 2$
- $\left(\frac{dy}{dt}\right) + y = kt. \quad \textit{The order is } 1$

- **Degree of Differential Equation**

The degree of the differential equation is the power of the highest order derivative, where the original equation is represented in the form of a polynomial equation in derivatives such as $y'$, $y''$, $y'''$, and so on.

Suppose $\left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right) + y = 0$ is a differential equation, so the degree of this equation here is 1. See some more examples here:

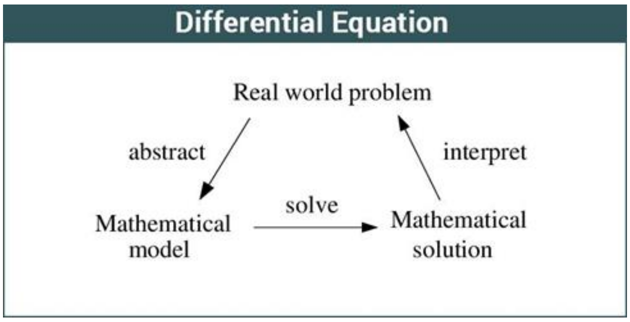
$$\frac{dy}{dx} + 1 = 0, \quad \textit{degree is } 1$$

$$(y''')^3 + 3y'' + 6y' - 12 = 0, \quad \textit{degree is } 3$$

- **Types of Differential Equations**

Differential equations can be divided into several types namely:

- Ordinary Differential Equations
- Partial Differential Equations
- Linear Differential Equations
- Nonlinear differential equations
- Homogeneous Differential Equations
- Nonhomogeneous Differential Equations




- **Ordinary Differential Equation**

An ordinary differential equation involves function and its derivatives. It contains only one independent variable and one or more of its derivatives with respect to the variable.

The order of ordinary differential equations is defined as the order of the highest derivative that occurs in the equation. The general form of n-th order ODE is given as

$$F(x, y, y', \dots, yn) = 0$$

- **Differential Equations Solutions**

A function that satisfies the given differential equation is called its solution. The solution that contains as many arbitrary constants as the order of the differential equation is called a general solution. The solution free from arbitrary constants is called a particular solution. There exist two methods to find the solution of the differential equation.

1. Separation of variables
2. Integrating factor

Note for reference only

**Separation of the variable** is done when the differential equation can be written in the form of $\frac{dy}{dx} = f(y)\, g(x)$ where $f$ is the function of $y$ only and $g$ is the function of $x$ only. Taking an initial condition, rewrite this problem as $\frac{1}{f(y)dy} = g(x)dx$ and then integrate on both sides.

**Integrating factor** technique is used when the differential equation is of the form $\frac{dy}{dx} + p(x)y = q(x)$ where $p$ and $q$ are both the functions of $x$ only.

Differential equations have several applications in different fields such as applied mathematics, science, and engineering. Apart from the technical applications, they are also used in solving many real life problems.

- **Linear Differential Equations Real World Example**

To understand Differential equations, let us consider this simple example. Have you ever thought about why a hot cup of coffee cools down when kept under normal conditions? According to Newton, cooling of a hot body is proportional to the temperature difference between its temperature T and the temperature $T_0$ of its surrounding. This statement in terms of mathematics can be written as:

$$\frac{dT}{dt} \propto (T - T_0)$$

This is the form of a linear differential equation.

Introducing a proportionality constant $k$, the above equation can be written as:

$$\frac{dT}{dt} = k(T - T_0)$$

Here, *T* is the temperature of the body and *t* is the time,

$T_0$ is the temperature of the surrounding,

$\frac{dT}{dt}$ is the rate of cooling of the body.

$$e.g.: \qquad \frac{dy}{dx} = 3x$$

Here, the differential equation contains a derivative that involves a variable (dependent variable, y) w.r.t another variable (independent variable, x). The types of differential equations are :

1. An ordinary differential equation contains one independent variable and its derivatives. It is frequently called ODE. The general definition of the ordinary differential equation is of the form: Given an *F*, a function of *x* and *y* and derivative of *y*, we have

$F(x, y, y' \ldots\ldots y\hat{}(n1)) = y\,(n)$ is an explicit ordinary differential equation of order n.

2. Partial differential equation that contains one or more independent variables.

The differential of a function of two variables, $f\ (x, y)$ , is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \qquad (3)$$

The terms like $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are called partial derivatives, because they are taken assuming that all other variables besides that in the denominator are constant.
For example, $\frac{\partial f}{\partial x}$ describes how $f$ changes as $x$ changes (holding $y$ constant), and $\frac{\partial f}{\partial y}$ describes how $f$ changes as $y$ changes (holding $x$ constant).

That partial and full derivatives are different can be illustrated by dividing Eq. (3) by the differential of $x$ to get

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} \qquad (4)$$

From Eq. (4) we see that the full derivative and the partial derivative are equivalent only if $x$ and $y$ are independent, so that $\frac{\partial y}{\partial x}$ is zero.

***WARNING!*** Partial derivatives are not like fractions. The numerators and denominators cannot be pulled apart or separated arbitrarily. Partial derivatives must be treated as a complete entity. So, you should **NEVER** pull them apart as shown below

$$\frac{\partial f}{\partial t} = axt^2 \quad \rightarrow \partial f = axt^2 \partial t \qquad \textit{Never do this}\,!$$

With a full derivative this is permissible, because is it composed of the ratio of two differentials. But there is no such thing as a *partial differential*, $\partial f$ .

- **Linear and nonlinear differential equations**

| Differential equation | Order | Degree | Linear | Nonlinear |
|---|---|---|---|---|
| 1. $y^2\frac{dx}{dy} + 2xy = 3x\sin y \rightarrow \frac{1}{3x}\frac{dx}{dy} = \frac{\sin y}{y^2} - \frac{2}{3}y$ | 1 | 1 | | ✓ $a_1(y)$ has $x$ as coef. |
| 2. $\ln x\left(\frac{d^3u}{dx^3}\right)^4 + \left(x^2+3\right)\left(\frac{d^2u}{dx^2}\right)^2 = \tan x$ | 3 | 4 | | ✓ derivative not first power |
| 3. $x\frac{dy}{dx} = 6y + 12x^4y^{2/3} \rightarrow 3x\frac{dz}{dx} - 6z = 12x^4$ $\boxed{\frac{dz}{dx} = \frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx}}$ | 1 | 1 | ✓ | |
| 4. $y^2\frac{dx}{dy} + 2x\cos y = xe^y \rightarrow \frac{y^2}{x}\frac{dx}{dy} = e^y - 2\cos y$ | 1 | 1 | | ✓ $a_1(y)$ has $x$ as coef. |
| 5. $(\tan 2x)(y''') + (x-23)(y'')^2 = \ln x$ | 3 | 1 | | ✓ derivative not first power |
| 6. $(1+x)\frac{dy}{dx} = 6y'' + 12x^4 \rightarrow 6y'' - (1+x)y' = -12x^4$ | 2 | 1 | ✓ | |

- **Solved problem**

**Question:**

Verify that the function $y = e - 3x$ is a solution to the differential equation

$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

**Solution:**

The function given is $y = e^{-3x}$ . We differentiate both sides of the equation with respect to *x*,

$$\frac{dy}{dx} = -3e^{-3x}$$

Now we again differentiate the above equation with respect to *x*,

$$\frac{d^2y}{dx^2} = 9e^{-3x}$$

We substitute the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and *y* in the differential equation given in the question,

On left hand side we get, $LHS = 9\ e^{-3x} + (-3\ e^{-3x}) - 6\ e^{-3x}$

$= 9e^{-3x} – 9e^{-3x} = 0$ (which is equal to RHS)

Therefore, the given function is a solution to the given differential equation.

- Practice

1. What is differential Equation?
2. Mention the various types of differential equations.
3. What is the order of the differential Equation?
4. What is the use of a differential equation?

## 1.2 The Chain Rule

If $x$ and $y$ are not independent, but depend on a third variable such as $s$ [i.e., $x(s)$ and $y(s)$], then the chain rule is

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad (5)$$

If $x$ and $y$ are on multiple variables such as $s$ and $t$ [i.e., $x(s,t)$ and $y(s,t)$], then the chain rule is

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \qquad (6)$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \qquad (7)$$

## 1.3 The product rule and the quotient rule

The product and quotient rules also apply to partial derivatives:

- The *product* rule

$$\frac{\partial}{\partial x}(uv) = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} \qquad (8)$$

- The *quotient* rule

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) \qquad (9)$$

## 1.4 Partial differentiation is commutative

Another important property of partial derivatives is that it doesn't matter in which order you take them. In other words

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} \qquad (10)$$

Multiple partial derivatives taken with respect to different variables are known as *mixed* partial derivative.

## 1.5 Other important identities

$$\left(\frac{\partial f}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial f}\right)_y} \quad ; \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{1}{\left(\frac{\partial y}{\partial f}\right)_x} \qquad (11)$$

If a function of two variables is constant, such as $f(x, y) = c$ , then its differential is equal to zero,

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0 \qquad (12)$$

In this case, *x* and *y* must be dependent on each other, because in order for *f* to be a constant, as *x* change *y* must also change. For example, think of the function

$$f(x, y) = x^2 + y = c \qquad (13)$$

Eq. (12) can be rearranged to

$$\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{dy} + \left(\frac{\partial f}{\partial y}\right)_x = 0 \qquad (14)$$

The derivative $\frac{dx}{dy}$ in Eq. (14) is actually a partial derivative with *f* held constant, so we can write

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial y}\right)_f + \left(\frac{\partial f}{\partial y}\right)_x = 0 \qquad (15)$$

which when rearranged leads to the identity

$$\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial f}\right)_x + \left(\frac{\partial x}{\partial y}\right)_f = -1 \qquad (16)$$

Eq. (16) is only true if the function *f* is constant, so that $df = 0$.

## 1.6 Integration of partial derivatives

Integration is the opposite or inverse operation of differentiation.

$$\int_a^b \frac{\partial f(s,t)}{\partial s} ds = f(b,t) - f(a,t) \qquad (17)$$

$$\int_a^b \frac{\partial f(s,t)}{\partial t} dt = f(s,b) - f(s,a) \qquad (18)$$

## 1.7 Differentiating an integral [Leibniz integral rule]

If an integration with respect to one variable is then differentiated with respect to a separate variable, such as

$$\frac{d}{dt}\int_{a}^{b} f(s,t,u)ds \qquad (19)$$

the result depends on whether or not the limits of integration, *a* and *b*, depend on *t*.

In general, if both *a* and *b*, depend on *t*, the result is

$$\frac{d}{dt}\int_{a(t,u)}^{b(t,u)} f(s,t,u)ds = \int_{a(t,u)}^{b(t,u)} \frac{\partial f(s,t,u)}{\partial t} ds + f(b,t,u)\frac{\partial b}{\partial t} - f(a,t,u)\frac{\partial a}{\partial t} \qquad (20)$$

If a does not depend on t then the term in Eq. (20) that involves $\frac{\partial a}{\partial t}$ will disappear. Likewise, if b does not depend on t, then the term containing $\frac{\partial b}{\partial t}$ will be zero.

Note for reference only

| Property | Logarithmic Rules | Natural Logarithmic Rules |
|---|---|---|
| Product rule | $log_a(xy) = log_a x + log_a y$ | $\ln ab = \ln a + \ln b$ |
| Quotient rule | $log_a\left(\frac{x}{y}\right) = log_a x - log_a y$ | $ln\frac{a}{b} = \ln a - ln\, b$ |
| Power rule | $log_a x^P = P\, log_a x$ | $ln\, m^p = P \ln m$ |
| Change of base rule | $log_a x = \frac{log_b x}{log_b a}$ | |
| Equality rule | If $log_a x = log_a y$ then $x = y$ | |
| Exponential & logarithmic | | $e^{\ln x} = x$ |
| Inverse property | | $\ln e^x = x \quad for\ x > 0$ |
| One to one property for exponents | | $if\ e^x = e^y \quad then\ x = y$ |
| One to one property for logarithms | | $if \ln x = \ln y\ then\ x = y$ |

Please visit:
https://byjus.com/maths/differential-equation/