Foundation of Mathematics 1

CHAPTER 3 RELATIONS ON SETS

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Chapter Three
Relations on Sets

3.1 Cartesian Product

Definition 3.1.1. A set A is called

(i) **finite** set if A contains finite number of element, say n, and denote that by $|A| = n$. The symbol $|A|$ is called the **cardinality** of A,

(ii) **infinite** set if A contains infinite number of elements.

Definition 3.1.2. The **Cartesian product (or cross product)** of A and B, denoted by $A \times B$, is the set $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

(1) The elements (a,b) of $A \times B$ are ordered pairs, a is called the **first coordinate (component)** of (a,b) and b is called the **second coordinate (component)** of (a,b).

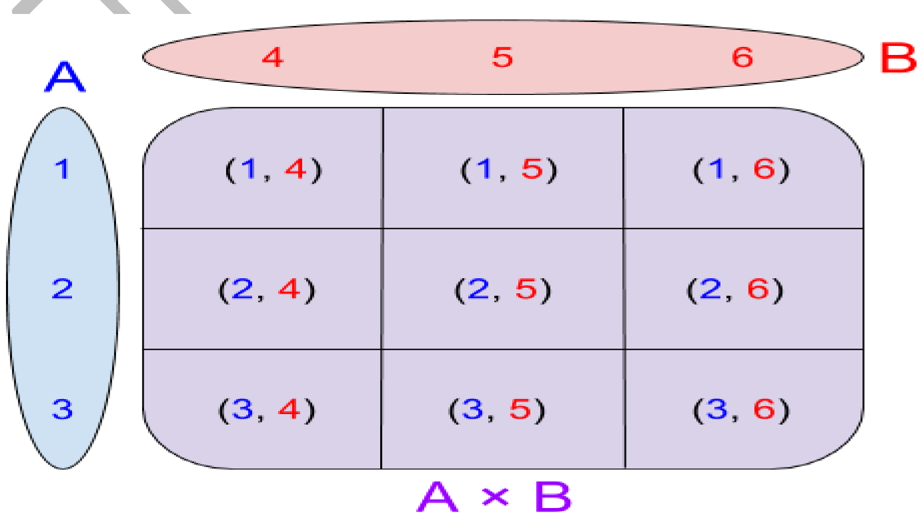
(2) For pairs $(a,b),(c,d)$ we have $(a,b) = (c,d) \Leftrightarrow a = c$ and $b = d$.

(3) The n-fold product of sets $A_1, A_2, \ldots, A_n$ is the set of n-tuples

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for all } 1 \leq i \leq n\}.$$

Example 3.1.3. Let $A = \{1,2,3\}$ and $B = \{4,5,6\}$.

(i) $A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$.



(ii) $B \times A = \{(4,1),(4,2),(4,3),(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}$.

Remark 3.1.4.

(i) For any set A, we have $A \times \emptyset = \emptyset$ (and $\emptyset \times A = \emptyset$) since, if $(a,b) \in A \times \emptyset$, then $a \in A$ and $b \in \emptyset$, impossible.

(ii) If $|A| = n$ and $|B| = m$, then $|A \times B| = nm$.

If A or B is infinite set then cross product $A \times B$ is infinite set.

(iii) Example 3.1.3 showed that $A \times B \neq B \times A$.

Theorem 3.1.5. For any sets A, B, C, D

(i) $A \times B = B \times A \Leftrightarrow A = B$,

(ii) if $A \subseteq B$, then $A \times C \subseteq B \times C$,

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,

(iv) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

(v) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$,

(vi) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$. The equialty may not hold.

(vii) $A \times (B - C) = (A \times B) - (A \times C)$.

Proof.

(i) The necessary condition. Let $A \times B = B \times A$. To prove $A = B$.

Let $x \in A \Longrightarrow (x, y) \in A \times B, \forall y \in B$. Def. of $\times$

$\Longrightarrow (x, y) \in B \times A$ By hypothesis

$\Leftrightarrow x \in B \wedge y \in A$ Def. of $\times$

(1) $\Longrightarrow x \in B \Longrightarrow A \subseteq B$ Def. of $\subseteq$

(2) By the same way we can prove that $B \subseteq A$.

Therefore, $A = B$ Inf. (1),(2).

The sufficient condition. Let $A = B$. To prove $A \times B = B \times A$.

$A \times B = A \times A = B \times A$ Hypothesis

(vii) $A \times (B - C) = (A \times B) - (A \times C)$.

$(x,y) \in A \times (B - C) \Leftrightarrow x \in A \wedge y \in (B - C)$ Def. of $\times$

$\Leftrightarrow x \in A \wedge(y \in B \wedge y \notin C)$ Def. of $-$

$\Leftrightarrow (x \in A \wedge x \in A) \wedge(y \in B \wedge y \notin C)$ Idempotent Law of $\wedge$

$\Leftrightarrow (x \in A \wedge y \in B)\wedge(x \in A \wedge y \notin C)$ Commut. and Assoc. Laws of $\wedge$

$\Leftrightarrow (x,y) \in (A \times B)\wedge (x,y) \notin (A \times C)$ Def. of $\times$

$\Leftrightarrow (x,y) \in (A \times B) - (A \times C)$ Def. of $-$

3.2 Relations

Definition 3.2.1. Any subset " R" of $A \times B$ is called a **relation between A and B** and denoted by $R(A,B)$. Any subset of $A \times A$ is called a **relation on A**.

In other words, if A is a set, any set of ordered pairs with components in A is a relation on A. Since a relation R on A is a subset of $A \times A$, it is an element of the power set of $A \times A$; that is, $R \in P(A \times A)$.

If R is a relation on A and $(x,y) \in R$, then we write $\boldsymbol{xRy}$, read as "x is in R-relation to y", or simply, x is in relation to y, if R is understood.

Example 3.2.2.

(i) Let $A = \{2,4,6,8\}$, and define the relation R on A by $(x,y) \in R$ iff x divides y. Then, $R =$ $\{(2,2),(2,4),(2,6),(2,8),(4,4),(4,8),(6,6),(8,8)\}$.

(ii) Let $A = \{0,3,5,8\}$, and define $R \subseteq A \times A$ by xRy iff x and y have the same remainder when divided 3.

$R = \{(0,0),(0,3),(3,0),(3,3),(5,5),(5,8),(8,5),(8,8)\}$.

Observe, that xRx for $x \in N$ and, whenever xRy then also yRx.

(iii) Let $A = \mathbb{R}$, and define the relation R on $\mathbb{R}$ by xRy iff $y = x^2$. Then R consists of all points on the parabola $y = x^2$.

(iv) Let $A = \mathbb{R}$, and define R on $\mathbb{R}$ by xRy iff $x \cdot y = 1$. Then R consists of all pairs $(x, \frac{1}{x})$, where x is non-zero real number.

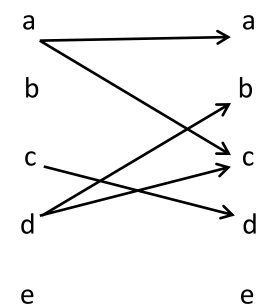
(v) Let $A = \{1,2,3\}$, and define R on A by xRy iff $x + y = 7$. Since the sum of two elements of A is at most 6, we see that xRy for no two elements of A; hence, $R = \emptyset$.

For small sets we can use a pictorial representation of a relation R on A: Sketch two copies of A and, if xRy then draw an arrow from the x in the left sketch to the y in the right sketch.

(vi) Let $A = \{a, b, c, d, e\}$, and consider the relation

$$R = \{(a,a),(a,c),(c,d),(d,b),(d,c)\}.$$

An arrow representation of R is given in Fig.



(vii) Let A be any set. Then the relation $R = \{(x,x): x \in A\} = I_A$ on A is called the **identity relation on *A***. Thus, in an identity relation, every element is related to itself only.

Definition 3.2.3. Let R be a relation between A and B. Then

(i) $\text{Dom}(R) = \{x \in A :$ There exists some $y \in B$ such that $(x,y) \in R\}$
is called the **domain of *R***.

(ii) $\text{Ran}(R) = \{y \in B:$ There exists some $x \in A$ such that $(x,y) \in R\}$

is called the **range of *R***.

Observe that $\text{Dom}(R)$ is subset of A, and $\text{Ran}(R)$ is subset of B.

Note: If R is a relation on A, then

$\text{Dom}(R) = \{x \in A : \text{There exists some } y \in A \text{ such that } (x,y) \in R\}$, and

$\text{Ran}(R) = \{y \in A: \text{There exists some } x \in A \text{ such that } (x,y) \in R\}$.

Observe that Dom(R) and Ran(R) are both subsets of A.

Example 3.2.4.

(i) Let A and R be as in Example 3.2.2(vi). Then

$\text{Dom}(R) = \{a,c,d\}$, $\text{Ran}(R) = \{a,b,c,d\}$.

(ii) Let $A = \mathbb{R}$, and define R by xRy iff $y = x^2$. Then

$\text{Dom}(R) = \mathbb{R}$, $\text{Ran}(R) = \{y \in \mathbb{R} : y \geq 0\}$.

(iii) Let $A = \{1,2,3,4,5,6\}$, and define R by xRy iff $x \lneqq y$ and x divides y; $R = \{(1,2),(1,3),\ldots,(1,6),(2,4),(2,6),(3,6)\}$, and Dom $(R) = \{1,2,3\}$, $\text{Ran}(R) = \{2,3,4,5,6\}$.

(iv) Let $A = \mathbb{R}$, and R be defined as $(x,y) \in R$ iff $x^2 + y^2 = 1$. Then

$(x,y) \in R$ iff (x,y) is on the unit circle with centre at the origin. So,

$$\text{Dom}(R) = \text{Ran}(R) = \{z \in \mathbb{R}: -1 \leq z \leq 1\}.$$

Definition 3.2.5. (Reflexive, Symmetric, antisymmetric and Transitive Relations)

Let R be a relation on a nonempty set A.

(i) R is **reflexive** if $(x,x) \in R$ for all $x \in A$.
(ii) R is **antisymmetric** if for all $x,y \in A$, $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$.
(iii) R is **transitive** if for all $x,y,z \in A$, $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in R$.
(iv) R is **symmetric** if whenever $(x,y) \in R$ then $(y,x) \in R$.

Definition 3.2.6.
(i) R is an **equivalence relation** on A, if R is reflexive, symmetric, and transitive. The set

$$[x] = \{y \in A: xRy\}$$

is called **equivalence class**. The set of all different equivalence classes A/R is called the **quotient set**.

(ii) R is a **partial order** on A(an **order** on A, or an **ordering** of A), if R is reflexive, antisymmetric, and transitive. We usually write $\leq$ for R; that is,

$$\boxed{x \leq y \text{ iff } xRy}.$$

(iii) If R is a **partial order** on A, then the element $a \in A$ is called **least element of *A* with respect to *R*** if and only if aRx for all $x \in A$.
(iv) If R is a **partial order** on A, then the element $a \in A$ is called **greatest element of *A* with respect to *R*** if and only if xRa for all $x \in A$.
(v) If R is a **partial order** on A, then the element $a \in A$ is called **minimal element of *A* with respect to *R*** if and only if xRa then $a = x$ for all $x \in A$.
(vi) If R is a **partial order** on A, then the element $a \in A$ is called **maximal element of *A* with respect to *R*** if and only if aRx then $a = x$ for all $x \in A$.

Example 3.2.7.
(i) The relation on the set of integers $\mathbb{Z}$ defined by

$$\boxed{(x, y) \in R \text{ if } x - y = 2k, \quad \text{for some } k \in \mathbb{Z}}$$

is an equivalence relation, and partitions the set integers into two equivalence classes, i.e., the even and odd integers.
If $y = 0$, then $[x] = \mathbb{Z}_e$. If $y = 1$, then $[x] = \mathbb{Z}_o$. $\mathbb{Z} = \mathbb{Z}_e \cup \mathbb{Z}_o$, $\mathbb{Z}/R = \{\mathbb{Z}_e, \mathbb{Z}_o\}$.

(ii) The inclusion relation $\subseteq$ is a partial order on power set $P(X)$ of a set X.
(iii) Let $A = \{3,6,7\}$, and

$$R_1 = \{(x, y) \in A \times A: x \leq y\},\ R_2 = \{(x, y) \in A \times A: x \geq y\}$$
$$R_3 = \{(x, y) \in A \times A: y \text{ divisible by } x\}$$

are relations defined on A.

$$R_1 = \{(3,3), (3,6), (3,7), (6,6), (6,7), (7,7)\},$$
$$R_2 = \{(3,3), (6,3), (6,6), (7,3), (7,6), (7,7)\}.$$
$$R_3 = \{(3,3), (3,6), (6,6), (7,7)\}.$$

R_1,R_2 and R_3 are partial orders on A.
(1) The least element of A with respect to R_1 is --------------.
(2) The least element of A with respect to R_2 is --------------.
(3) The greatest element of A with respect to R_1 is --------------.
(4) The greatest element of A with respect to R_2 is --------------.

(5) A has no least and greatest element with respect to R_3 since, --------------.
(6) The maximal element of A with respect to R_3 is --------------.
(7) The minimal element of A with respect to R_3 is --------------.
(iv) Let $X = \{1,2,4,7\}$, $K = \{\{1,2\},\{4,7\},\{1,2,4\},X\}$ and

$$R_1 = \{(A,B) \in K \times K : A \subseteq B\},$$
$$R_2 = \{(A,B) \in K \times K : A \supseteq B\},$$

are relations defined on K.

$$R_1 = \begin{matrix} (\{1,2\},\{1,2\}), & (\{1,2\},\{1,2,4\}), & (\{1,2\},X), \\ (\{4,7\},\{4,7\}), & (\{4,7\},X), & \\ (\{1,2,4\},\{1,2,4\}), & (\{1,2,4\},X), & \\ (X,X) & & \end{matrix}$$

$$R_2 = \begin{matrix} (\{1,2\},\{1,2\}), & & & \\ (\{4,7\},\{4,7\}), & & & \\ (\{1,2,4\},\{1,2\}), & (\{1,2,4\},\{1,2,4\}), & & \\ (X,\{1,2\}), & (X,\{4,7\}), & (X,\{1,2,4\}), & (X,X) \end{matrix}$$

R_1 and R_2 are partial orders on K.
(1) K has no least element with respect to R_1 since, --------------.
(2) The greatest element of K with respect to R_1 is --------------.
(3) The least element of K with respect to to R_2 is --------------.
(4) K has no greatest element with respect to R_2 since, --------------.
(5) The minimal elements of K with respect to R_1 are --------------.
(6) The maximal element of K with respect to R_1 is --------------.
(7) The minimal element of K with respect to R_2 is --------------.
(8) The maximal element of K with respect to R_2 is --------------.

Remark 3.2.8.
(i) Every greatest (least) element is maximal (minimal). The converse is not true.
(ii) The greatest (least) element if exist, it is unique.
(iii) Every finite partially ordered set has maximal (minimal) element.
Properties of equivalence classes
(iv) For all $a \in X$, $a \in [a]$.
(v) $aRb \Leftrightarrow [a] = [b]$.
(vi) $[a] = [b] \Leftrightarrow (a,b) \in R \Leftrightarrow aRb$.

(vii) $[a] \cap [b] \neq \emptyset \Leftrightarrow [a] = [b]$.
(viii) $[a] \cap [b] = \emptyset \Leftrightarrow [a] \neq [b]$.
(ix) For all $a \in X$, $[a] \in X/R$ but $[a] \subseteq X$.
Definition 3.2.9. R is **a totally order** on A if R is a partial order, and xRy or yRx for all $x, y \in A$; that is, if any two elements of A are comparable with respect to R. Then we call the pair $(A, \leq)$ a **totally order set** or a **chain**.

Example 3.2.10.

(i) Let $A = \{2,3,4,5,6\}$, and R a relation on A defined as the usual $\leq$ relation on $\mathbb{N}$, i.e. aRb iff $a \leq b$. Then R is a **totally order** on A.

(ii) Let "$/$" be a relation on $\mathbb{N}$ defined as follows:

$$a/b \text{ iff } a \text{ divides } b.$$

To show that / is a partial order we have to show the three defining properties of a partial order relation:
Reflexive: Since every natural number a, $a = a.1$; that is, a is a divisor of itself, so we have a/a for all $a \in A$.
Antisymmetric: If a divides b, then we have $b = ka$ for some k in $\mathbb{N}$. If b divides a, then $a = tb$ for some t in $\mathbb{N}$. So, $a = kta$, thus $kt = 1$; that is, $k = 1$ and $a = 1$. Therefore, $a = b$.

Transitive: If a divides b, then we have $b = ka$ for some k in $\mathbb{N}$, and if b divides c, then we have $c = tb$ for some t in $\mathbb{N}$. Thus, $c = t(ka) = (tk)a$. Therefore, a divides c.

Thus, / is a partial order on N.

The relation "$/$" is not totally order since $(3,4) \notin /$.

(iii) Let $A = \{x, y\}$ and define $\leq$ on the power set $P(A) = \{\emptyset, \{x\}, \{y\}, A\}$ by

$$s \leq t \text{ iff } s \text{ is a subset of } t.$$

This gives us the following relation:

$\emptyset \leq \emptyset, \emptyset \leq \{x\}, \emptyset \leq \{y\}, \emptyset \leq \{x,y\} = A, \{x\} \leq \{x\}, \{x\} \leq \{x,y\}, \{y\} \leq \{y\}, \{y\} \leq \{x,y\}, \{x,y\} \leq \{x,y\}$.

The relation "$\leq$" is not totally order since $(\{x\},\{y\}) \notin \leq$.

Exercise 3.2.11.

Let $A = \{1, 2, \dots, 10\}$ and define the relation R on A by xRy iff x is a multiple of y. Show that R is a partial order on A. (Hint: $R = \{(ny, y)$: for some $n \in \mathbb{Z}$ and $y \in A\})$

Definition 3.2.12. (Inverse of a Relation)
Suppose $R \subseteq A \times B$ is a relation between A and B then the inverse relation $R^{-1} \subseteq B \times A$ is defined as the relation between B and A and is given by

$$bR^{-1}a \quad \text{if and only if} \quad aRb.$$

That is, $R^{-1} = \{(b,a) \in B \times A : (a,b) \in R\}$.

Example 3.2.13. Let R be the relation between $\mathbb{Z}$ and $\mathbb{Z}^+$ defined by

$$mRn \text{ if and only if } m^2 = n.$$

Then

$$R = \{(m,n) \in \mathbb{Z} \times \mathbb{Z}^+ : m^2 = n\} = \{(m, m^2) \in \mathbb{Z} \times \mathbb{Z}^+\},$$

and

$$R^{-1} = \{(n,m) \in \mathbb{Z}^+ \times \mathbb{Z} : m^2 = n\} = \{(m^2, m) \in \mathbb{Z}^+ \times \mathbb{Z}\}.$$

For example, $-3\,R\,9$, $-4\,R\,16$, $16\,R^{-1}\,4$, $9\,R^{-1}\,3$, etc.

Remark 3.2.14. If R is partial order relation on $A \neq \emptyset$, then

(i) R^{-1} is also partial order relation on A.

(ii) $(R^{-1})^{-1} = R$.

(iii) Dom(R^{-1}) =Ran(R) and Ran(R^{-1}) =Dom(R).

Proof. (i)

(1) Reflexive. Let $x \in A$.

$\Longrightarrow (x,x) \in R$ (Reflexivity of A) $\Longrightarrow (x,x) \in R^{-1}$ Def of R^{-1}

(2) Anti-symmetric. Let $(x,y) \in R^{-1}$ and $(y,x) \in R^{-1}$. To prove $x = y$.

$\Longrightarrow (y,x) \in R \wedge (x,y) \in R$ Def of R^{-1}

$\Longrightarrow y = x$ Since R is antisymmetric

(3) Transitive. Let $(x, y) \in R^{-1}$ and $(y, z) \in R^{-1}$. To prove $(x, z) \in R^{-1}$.

$\Longrightarrow (y, x) \in R \wedge (z, y) \in R$ Def of R^{-1}

$\Longrightarrow (z, y) \in R \wedge (y, x) \in R$ Commut. Law of $\wedge$

$\Longrightarrow (z, x) \in R$ Since R is transitive

$\Longrightarrow (x, z) \in R^{-1}$ Def of R^{-1}

Definition 3.2.15. (Partitions)
Let A be a set and let $A_1, A_2, \ldots, A_n$ be subsets of A such
(i) $A_i \neq \emptyset$ for all i,
(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$,
(iii) $A = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$. Then the sets A_i partition the set A and these sets are called the **classes of the partition.**

Remark 3.2.16. An equivalence relation on X leads to a partition of X, and **vice versa** for every partition of X there is a corresponding equivalence relation.

Proof:

(a) Let R be an equivalence relation on X.

1- $\forall a \in X$, $a \in [a]$ Def. of equ. Class

2- $\exists [b] \in X/R$ such that $[b] = [a]$ Since X/R contains all diff. classes

3- $X = \bigcup_{a \in X}\{a\} \subseteq \bigcup_{a \in X}[a] \subseteq \bigcup_{a \in [b]}[b] \subseteq X \Longrightarrow X = \bigcup_{[b] \in X/R}[b]$.

4- $[b] \cap [a] = \emptyset$, for all $[b], [a] \in X/R$ Def. of X/R

5- R is partition of X Inf.(3),(4)

(b) Let (i) $A_i \neq \emptyset$ for all i, $A_i \subseteq X$
(ii) $A_i \cap A_j = \emptyset$ if $i \neq j$,
(iii) $X = \bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$.

Define R (relation) on X by $aRb \Leftrightarrow$ if $\exists A_i$ such that $a, b \in A_i$.

This relation is an equivalence relation on X.

Definition 3.2.17. (The Composition of Two Relations)

The composition of two relations $R_1(A,B)$ and $R_2(B,C)$ is given by $R_2\ o\ R_1$ where $(a,c) \in R_2\ o\ R_1$ if and only if there exists $b \in B$ such that $(a,b) \in R_1$ and $(b,c) \in R_2$. That is,

$$R_2\ o\ R_1 = \{(a,c) \in A \times C | \exists\ b \in B \text{ such that} (a,b) \in R_1 \text{ and } (b,c) \in R_2\}$$

Remark 3.2.18. Let $R_1(A,B)$, $R_2(B,C)$ and $R_3(C,D)$ are relations. Then,

(i) $(R_3\ o\ R_2)\ o\ R_1 = R_3\ o\ (R_2\ o\ R_1)$.

(ii) $(R_2\ o\ R_1)^{-1} = {R_1}^{-1} o\ {R_2}^{-1}$.

(iii) Let $R^{-1} = \{(b,a)|(a,b) \in R\} \subseteq B \times A$. Then

$$(a,b) \in R\ o\ R^{-1} \Leftrightarrow (b,a) \in R\ o\ R^{-1}, \text{ for every for every } a, b \text{ in } B.$$

Proof. Exercise.

Example 3.2.19.

Let sets $A = \{a,b,c\}, B = \{d,e,f\}, C = \{g,h,i\}$ and relations

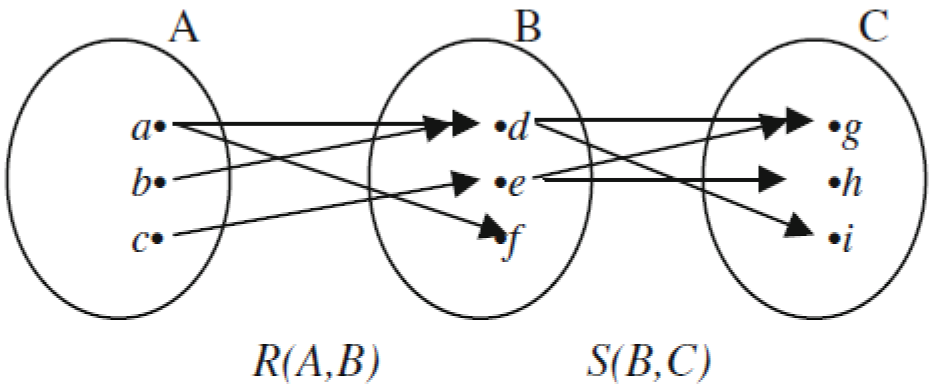
$$R(A,B) = \{(a,d),(a,f),(b,d),(c,e)\}$$

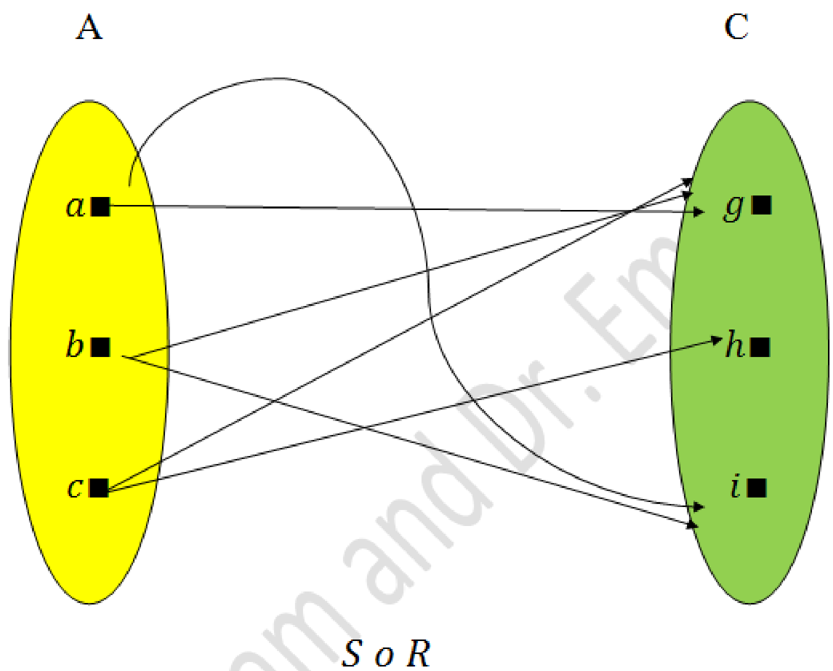
and

$$S(B,C) = \{(d,g),(d,i),(e,g),(e,h)\}.$$

Then we graph these relations and show how to determine the composition pictorially S o R is determined by choosing $x \in A$ and $y \in C$ and checking if there is a route from x to y in the graph. If so, we join x to y in $S\ o\ R$.

$$S\ o\ R = \{(a,g),(a,i),(b,g),(b,i),(c,g),(c,h)\}.$$





For example, if we consider a and g we see that there is a path from a to d and from d to g and therefore (a, g) is in the composition of S and R.

Definition 3.2.19. Union and Intersection of Relations

(i) The union of two relations $R_1(A, B)$ and $R_2\ (A, B)$ is subset of $A \times B$ and defined as

$$(a, b) \in R_1 \cup R_2 \text{ if and only if } (a, b) \in R_1 \text{ or } (a, b) \in R_2.$$

(ii) The intersection of two relations $R_1(A,B)$ and $R_2\ (A,B)$ is subset of $A \times B$ and defined as

$$(a,b) \in R_1 \cap R_2 \text{ if and only if } (a,b) \in R_1 \text{ and } (a,b) \in R_2.$$

Remark 3.2.20.
(i) The relation R_1 is a subset of R_2 ($R_1 \subseteq R_2$) if whenever $(a,b) \in R_1$ then $(a,b) \in R_2$.
(ii) The intersection of two equivalence relations R_2, R_1 on a set X is also equivalence relation on X.
(iii) In general, the union of two equivalence relations R_1, R_2 on a set X need not to be an equivalence relation on X.
Proof. Exercise.

Example 3.2.21. Let $X = \{a,b,c\}$. Define two relations on X as follows:

$$R_1(X,X) = \{(a,a),(b,b),(c,c),(a,b),(b,a)\},$$

$$R_2(X,X) = \{(a,a),(b,b),(c,c),(a,c),(c,a)\}.$$

Let $R = R_1 \cup R_2$. Here, R is not an equivalence relation on X since it is not transitive relation, because (b,a) and $(a,c) \in R$ but $(b,c) \notin R$.