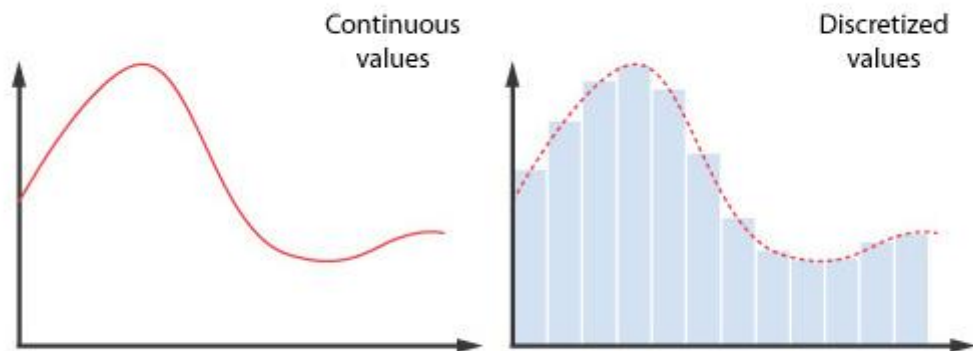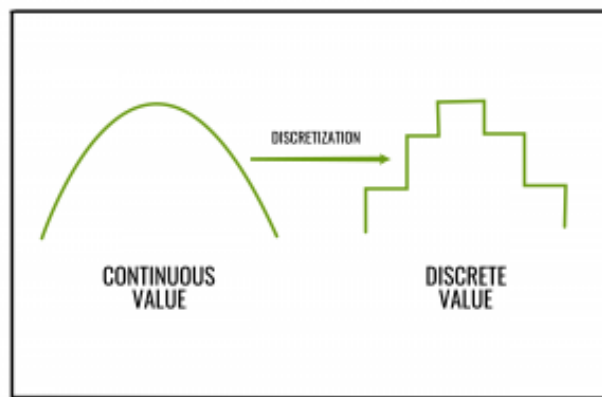# Lecture (12)

# Discretization Error and Convergence

## 12.1 Discretization Error

**Discretization** is the process of transferring continuous functions, models, variables, and equations into discrete counterparts. This process is usually carried out as a first step toward making them suitable for numerical evaluation and implementation on digital computers.



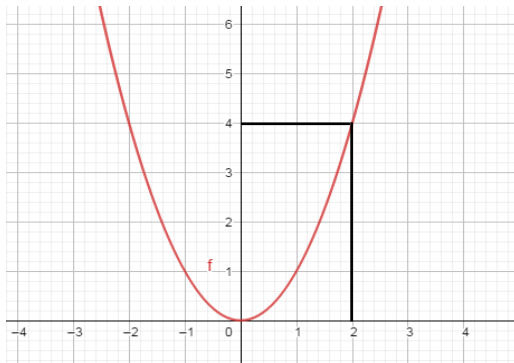Ex: Solve the following integration analytically and numerically

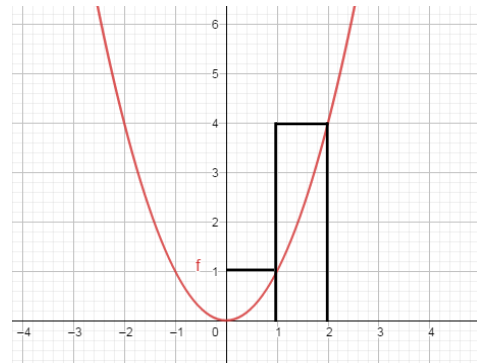$$\int f(x)dx = \int x^2\, dx$$

Sol. (1) Analytically:

$$\int f(x)dx = \int x^2\, dx = \frac{x^3}{3} + C$$
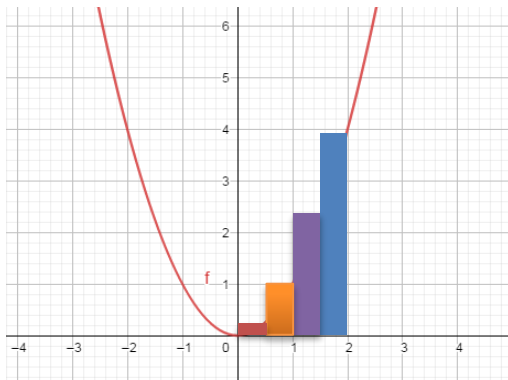
$$at\ x = 2\ the\ solution\ is \frac{2^3}{3} = 2.666$$

(2) Numerically (Area under the curve)



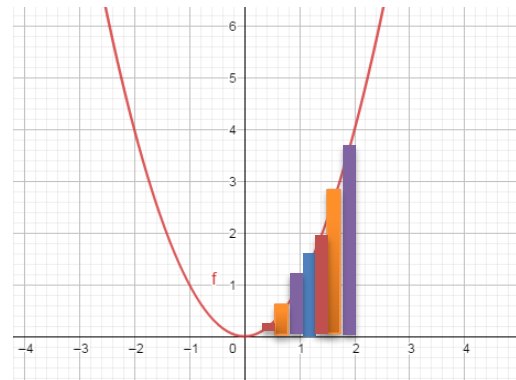A= 8                                          A= 5



A= 3.77                                      A= 2.97

As we can see, the solution is approaching the exact value (the analytical solution) as we adding more fine rectangles (less $\Delta x$)

## 12.2 Truncation Error



## Truncation Errors

Truncation errors are the errors that result from using an approximation in place of an exact mathematical procedure.

Approximation                              Truncation Errors

$$e^x = \underbrace{\left|1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!}\right.}_{\text{Exact mathematical formulation}} + \left.\frac{x^{n+1}}{(n+1)!} + ...\right|$$

Exact mathematical formulation

**Analytical solutions** are logical procedures that yield an **exact solution**.

**Numerical solutions** come from discrete equations that result in approximate solutions.

The truncation error can be made as small as we wish by making $\Delta x$ and $\Delta t$ smaller and smaller, as long as the system is consistent.

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**CONSISTENCY**: A **finite difference approximation** is considered consistent if by reducing the mesh and time step size, the **truncation error** terms could be made to approach zero. In that case, the solution to the difference equation would approach the true solution to the PDE.

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There is another source of errors is the infinite accuracy of computers. This round off error is a feature of the computers and the program. Sometimes this error causes problems but not often, we will not worry about it for now.

If we take into account acceptable levels of truncation error and round off error, we should take into account the error in solving the solution in discrete equation, that is, the difference between the solution of discrete equation and the solution of continuous differential equation، i.e.

$$A(i,j) - A(i\Delta x, j\Delta t)$$

This error is called the Discretization Error. Unfortunately, a decrease in the truncation error does not necessarily confirm a decrease in the discretization error. It is even possible that the decrease in the truncation error to be accompanied by an increase in the discretization error!

We will now analyze the changes of the discretization error with smoothing of the grid (that is, as $\Delta$x and $\Delta t$ get closer to zero). If the discretization error approaches zero, then we say that the solution converges.

Figure (12.1) illustrates a situation in which the solution does not converge with further smoothing of the mesh. The diagonal line in the figure shows the feature

along which A is spread, that is, A is constant along the line. This line represents the exact solution (the analytical solution).

To work on numerical approximation for this solution, we first choose Δx and Δt so that the grid points are the bold points in the figure. The set of grid points with A values and on which $A_{i,j}$ is based is called the domain of dependence. The shaded area in the figure shows the dependency for an upstream scheme described by the equation

$$\frac{A_{i,j+1} - A_{i,j}}{\Delta t} + c\left(\frac{A_{i,j} - A_{i-1,j}}{\Delta x}\right) = 0 \qquad (12.1)$$


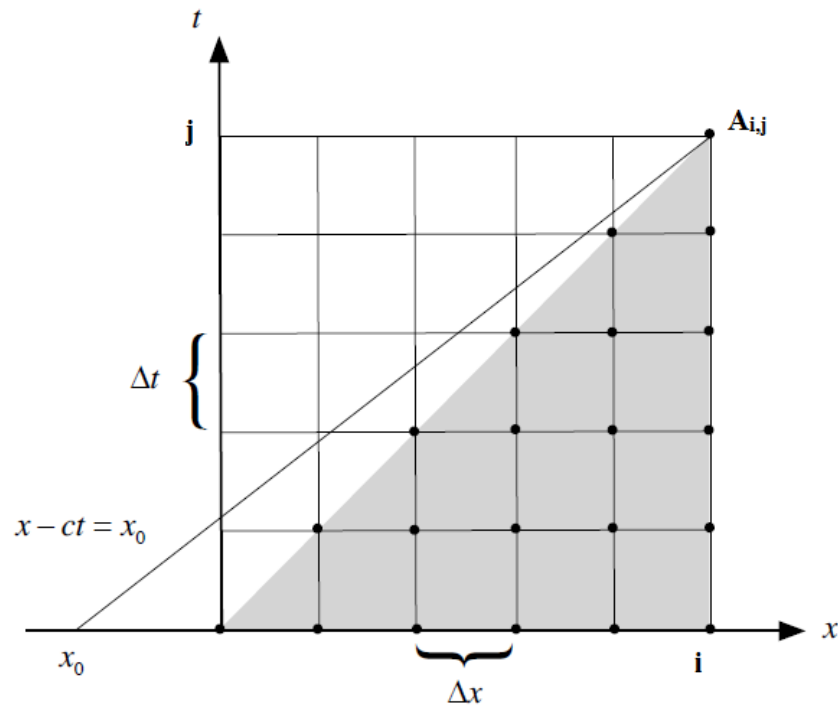
Figure (12.1) the shaded area in the figure represents a domain of dependency for an upstream scheme in the point $(i\Delta x, j\Delta t)$.

We can increase the accuracy of the scheme by bisections of $\Delta x$ و $\Delta t$ but the dependence domain does not change, regardless of the magnitude of the grid density as long as the ratio cΔt / Δx remains constant. This is evidence that cΔt / Δx is an important quantity.

Suppose that the line that passes through the point (iΔx, jΔt) is $x - ct = x_0$, where $x_0$ is a constant quantity, which does not fall within the domain of dependence. This situation is shown in Figure (12.1). In general, it is not hoped to obtain a smaller

discretization error, no matter how much the values of $\Delta x$ and $\Delta t$ change as long as the quantity $\frac{c\Delta t}{\Delta x}$ is unchanged, because the true solution (the true numerical solution) depends only on the initial value of A in the individual point $(x_0, 0)$ that cannot influence $A_{i,j}$.

You can change $A(x_0, 0)$ [and hence the exact solution $A(i\Delta x, j\Delta t)$], but the computed solution $A_{i,j}$ will remain the same. In such a case, the error in the solution will not be diminished by the smoothing of the grid. This shows that if the value of c is such that $x_0$ falls outside the domain of dependency, then it is not possible for the solution of the finite difference equation (numerical solution) to get close to the solution of the differential equation (analytical), whether the grid becomes smooth or not. The finite difference equation converges to the differential equation, but the solution to the differential equation does not converge to the solution of the differential equation. The truncation error goes to zero but the discretization error does not.

The above discussion shows that:

$$0 \leq \frac{c\Delta t}{\Delta x} \leq 1 \qquad (12.2)$$

is a necessary condition for convergence in upstream scheme

Note that if c is negative (giving the downstream scheme)

The distinguished line is outside the domain of dependence in the figure. Actually, if $< 0$, we can get:

$$\frac{A_{i,j+1} - A_{i,j}}{\Delta t} + c\left(\frac{A_{i,j+1} - A_i}{\Delta x}\right) = 0 \qquad (12.3)$$

Instead of the upstream scheme equation (12.1).

It is a measure of the accuracy with which you approximate the terms of the differential equation. The dissociation error measures the accuracy with which the solution of the differential equation approximates. Reducing the cutoff error to acceptable levels is usually easy, but reducing the separation error is more difficult.

In summary: the truncation error measures the approximation accuracy of the differential operator. It is a measure for the accuracy the terms of the differential equation are approximated. Reducing the truncation error to acceptable levels is usually easy, but reducing the discretization error is more difficult.