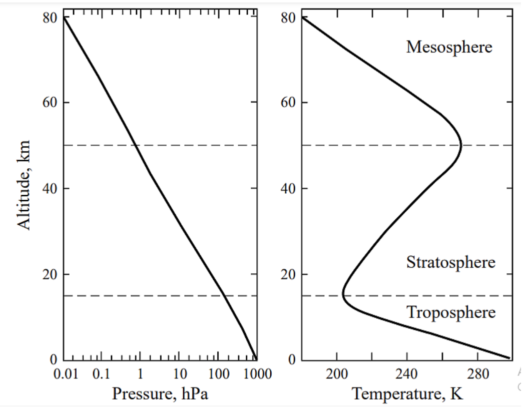
Atmospheric Thermodynamics

Lecture 5: Vertical Thermodynamic Structure of the Atmosphere

5.1 The Basic Change of Pressure and Temperature with Altitude

- In atmospheric physics, we deal with cloud formation and precipitation for which moist air has to ascent, cool and condense to form clouds. Hence vertical movement of moist air is very much essential for the cloud formation.
- Vertical change of temperature determine stability and mixing of air pollutants.
- As altitude increases, the amount of gas molecules in the air decreases- the air becomes less dense than air nearer to sea level. Hence, the air pressure and density decreases with height.
- As altitude increases, temperature decreases. Various factors are responsible for this, including air pressure and water-vapor content. Every 100 meters, the temperature drops by an average of 0.65°C. If the air is very dry, such as in an area of high pressure, the air can cool by almost 1°C per 100 meters.



5.2 The Standard Atmosphere

- Air is ideal dry gas.
- The physical constants are:
  - Mean Sea Level (MSL) of mean molecular weight is 28.966 kg mol^{-1}
  - MSL variables are: atmospheric pressure P_0=1013.25 mb, air temperature T_0=288.15 K , air density $\rho = 1.25\ kg\ m^{-3}$
  - Universal gas constant $R = 8.314\ J\ mol^{-1} K^{-1}$

5.3 Hydrostatic Equilibrium

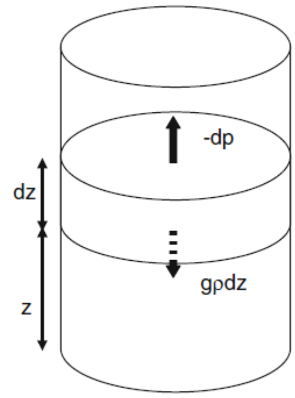
Atmospheric air pressure at any height in the atmosphere is due to the force per unit area exerted by the weight of all of the air lying above that height. Consequently, atmospheric pressure decreases with increasing height above the ground.

Newton's law requires that the upward force acting on thin a layer of air from the decrease of pressure with height is generally closely balanced by the downward force due gravity (as in the figure). The hydrostatic equation is then:

$$\frac{\partial p}{\partial z} = -g\rho \qquad (5.1)$$

Typically, deviations from the hydrostatic balance occur locally, e.g. in updrafts and downdrafts or when the air hits a small obstacle. In contrast to the hydrostatic balance, then an air particle undergoes acceleration:

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{dp}{dz} - g \qquad (5.2)$$



The atmosphere is in hydrostatic balance essentially everywhere except in core regions of significant storms such as hurricanes and thunderstorms.
another expression of (5.1) is:

$$dP + \rho\, dz\, g = 0 \qquad (5.3)$$

The equation above is the desired basic equation that can be used for obtaining the pressure at any height.

- For **incompressible fluids**, density is independent of pressure. Integrating the Equation above, we get

$$dP + \rho\, dz\, g = 0$$

$$\int dP + \rho\, g \int dz\ = 0$$

$$P + z\,\rho g = constant \qquad (5.4)$$

The equation above shows that the pressure reaches its maximum level at the base. This is true for the column or container of the fluid. The pressure decreases as we move up the column.

Consider that the pressure at the bottom of the column is p_1, where $z = 0$, and the pressure at any height z above the base is p_2 such that $p_1\ >\ p_2$, then

$$\int_{p_2}^{p_1} dp = \rho\, g \int_0^z dz \qquad (5.5)$$

Integrating the above equation, we get

$$(p_1 - p_2) = \rho\, g\, z \qquad (5.6)$$

where p_1 and p_2 are expressed in N/m^2 , ρ is in kg/m^3 , z is in m in SI Units.

The above equation helps in obtaining the pressure difference in a fluid. This can be done by measuring the height of the vertical column of the fluid between any two points.

- For **compressible fluids**, density changes with pressure.

For an ideal gas, the density is given by the following relation:

$$\rho = \frac{pM}{RT} \qquad (5.7)$$

Here,

p represents pressure.

M indicates the molecular weight of the gas.

R stands for the universal gas constant (=8.3145 J mol^{-1} K^{-1}).

T signifies the temperature.

Putting the value of ρ from the ideal gas law equation into the equation of Hydrostatic equilibrium,

$$dp + g\frac{pM}{RT}\,dz = 0$$

Rearranging Equation,

$$\frac{dp}{p} + g\frac{M}{RT}\,dz = 0$$

Integrating Equation, we get

$$ln\,p + g\frac{M}{RT}\,dz = 0$$

Integrating the above equation between two heights z_1 and z_2 where the pressures acting are p_1 and p_2, we get

$$ln\,\frac{p_2}{p_1} = -g\frac{M\,(z_2 - z_1)}{RT}$$

$$\frac{p_2}{p_1} = e^{\frac{-\,g\,M(z_2 - z_1)}{RT}} \qquad (5.8)$$

The equation above is known as the **Barometric Equation**. It gives us the idea of pressure distribution within an ideal gas for isothermal conditions.

Now, by writing eq. (5.8) using the notation p_0 for the pressure at zero altitude, and the p for the pressure at any upper altitude (h),

$$p = p_0\,e^{\frac{-M\,g\,h}{RT}} = p_0\,e^{\frac{-\,h}{H}} \qquad (5.9)$$

where H is the Scale Height ($H = \frac{RT}{Mg} = \frac{N_A k\,T}{M\,g}$, N_A is Avogadro's number= 6.023×10^{23}, k is Boltzmann's constant = $1.38 \times 10^{-23}\,J/K$).

It was found that the scale height at sea level is between 4 and 8 km depending on the temperature.

The scale height is a very useful concept because it is the height at which the atmosphere would extend if it were all compressed into one of constant density. At this height the real atmosphere has a density of 1/e or 0.37 of its density at the reference level. The scale height is also a measure of the atmospheric density gradient - a lower scale height implies a higher gradient.

Homework: Calculate the scale height of the atmosphere? (Consider N_2 gas only)

5.4 Special Hypothetical Atmospheres

A. Homogeneous Atmosphere

It is A hypothetical atmosphere in which the density is constant with height.

$\rho = \rho_o = constant$ (where ρ_o is the air density at the surface)

From the hydrostatic equation $\frac{dp}{dz} = -g\rho_o$

$$\int_{p_o}^{p} dp = -g\rho_o \int_0^z dz$$

$$p - p_o = -g\ \rho_o\ z$$

$$p = p_o - g\rho_o z \qquad (5.10)$$

Where p_o is pressure at $z = 0$

The homogeneous atmosphere has a finite height H,

when p=0 (at the top of the atmosphere), z=H

$$0 = p_o - \rho_o g H$$

$$\therefore H = \frac{p_o}{\rho_o g}$$

from the hydrostatic equation and the equation of state ($p_o = \rho_o\ R\ T_o$) we get:

$$H = \frac{\rho_o\ R\ T_o}{\rho_o\ g}$$

At $T_o = 283^o K$, $R = 287$, $g = 9.8\ ms^{-2} \Rightarrow H \approx 8000\ m$

Homework: solve for T_o = 293, 300, 310, 320, 330 ^oK

We may define a temperature in the homogeneous atmosphere from gas equation:

$$p = \rho_o R\ T$$

$$T = \frac{p}{\rho_o R} \qquad (5.11)$$

Put eqn. (5.10) in eqn. (5.11)

$$T = \frac{p_o - \rho_o\ g\ z}{\rho_o\ R} \quad \Rightarrow \quad T = \frac{p_o}{\rho_o R} - \frac{\rho_o\ g\ z}{\rho_o R}$$

$$\boxed{T = T_o - \frac{g}{R} z \qquad (5.12)}$$

This equation shows that T decreases linearly with height in a homogeneous atmosphere.

Question: From the atmospheric model (homogeneous atmosphere) show that the lapse rate $\gamma = \frac{dT}{dZ} = -\frac{g}{R} = -3.4\ ^oK/100m$ (such an atmosphere is known as the auto convective lapse rate)

B. The Isothermal Atmosphere

In this model we have $T = T_o = const.$ (where T_o is the temperature at the surface)

From the hydrostatic Equation we get:

$$dp = -\rho g\, dz$$

Recall that $\rho = \frac{p}{RT_o}$

$$dp = \frac{p}{RT_o} g\, dz$$

$$\int_{p_o}^{p} \frac{dp}{p} = -\frac{g}{RT_o} \int_0^z dz$$

$$\ln \frac{p}{p_o} = -\frac{g}{RT_o} z$$

Taking exponential to both sides

$$\frac{p}{p_o} = e^{-\frac{g}{RT_o} z}$$

This equation shows that the isothermal atmosphere is of infinite extent because $p \to 0$ when $z \to \infty$

$$\boxed{p = p_o e^{-\frac{g}{RT_o} Z} \qquad (5.13)}$$

The scale height for an isothermal atmosphere is often defined as the height at which the pressure has decreased to e^{-1} of the surface pressure.

$$z = H_s$$
$$p = p_o\, e^{-\frac{g}{RT_o} H_s}$$
$$p = p_o\, e^{-1}$$
$$p_o e^{-\frac{g}{R\,T_o} H_s} = p_o\, e^{-1}$$
$$-\frac{g}{RT_o} H_s = -1$$

$$\therefore H_s = \frac{R\, T_o}{g} = 8000\, m$$

Or, that the scale height is equal to the height of the homogeneous atmosphere having the same surface temperature as the isothermal atmosphere.

The density in the isothermal atmosphere can be calculated from gas equation

$p_o = \rho_o \, R \, T_o$, $p = \rho \, R \, T_o$:

$$p = p_o e^{-\frac{g}{RT_o}Z}$$

$$\rho \, R \, T_o = \rho_o \, R \, T_o \, e^{-\frac{g}{RT_o}z}$$

$$\therefore \rho = \rho_o \, e^{-\frac{g}{RT_o}z}$$

Problem 1: Show that a homogeneous atmosphere (density independent of height) has a finite height that depends only on the temperature at the lower boundary. Compute the height of a homogeneous atmosphere with surface temperature T_0 = 273K and surface pressure 1000 hPa. (Use the ideal gas law and hydrostatic balance.)

Problem 2: Show that in an atmosphere with uniform lapse rate γ (where $\gamma \equiv -\frac{dT}{dz}$) the geopotential height at pressure level p_1 is given by

$$Z = \frac{T_0}{\gamma}\left[1 - \left(\frac{p_0}{p_1}\right)^{-R\gamma/g}\right]$$

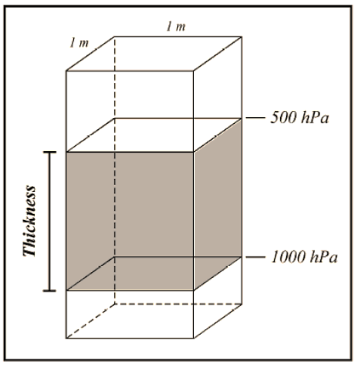
where T_0 and p_0 are the sea level temperature and pressure, respectively.

(Hint: Use the hydrostatic equation and the ideal gas law.)

5.5 Hypsometric Equation

Consider a column of atmosphere that is 1 m by 1 m in area and extends from sea level to space.

Let's isolate the part of this column that extends between the 1000 hPa surface and the 500 hPa surface. How much mass is in the column?



$$Mass = (1000 - 500)hPa \times \left(\frac{100 \, Nm^{-2}}{hPa}\right) \times 1 \, m \times \left(\frac{1}{9.81 \, m \, s^{-2}}\right) = 510204 \, kg$$

From the hydrostatic equation

$$\frac{dp}{dz} = -g\,\rho$$

From the ideal gas law

$p = \rho\, R\, T$

Substitute ideal gas law into hydrostatic equation

$$dz = -\frac{R\,T}{g}\,\frac{dp}{p}$$

Integrate this equation between two levels (p_2, z_2) and (p_1, z_1)

$$\int_{z_1}^{z_2} dz = -\int_{p_1}^{p_2} \frac{R\,T}{g}\,\frac{dp}{p}$$

$$\int_{z_1}^{z_2} dz = \int_{p_2}^{p_1} \frac{R\,T}{g}\; d\ln p$$

Problem: T varies with altitude. To perform the integral on the right we have to consider the pressure weighted column average temperature given by:

$$\boxed{z_2 - z_1 = \frac{R\,\bar{T}}{g} \ln \frac{p_1}{p_2}}$$

This equation is called the Hypsometric Equation. It relates the thickness of a layer of air between two pressure levels to the average virtual temperature of the layer.

5.6 Geopotential Height

We can express the hypsometric (and hydrostatic) equation in terms of a quantity called the geopotential height.

Geopotential (∅): Work (energy) required to raise a unit mass a distance dz above sea level.
$d\emptyset = g\,dz$

$$\boxed{\Delta\emptyset = R\bar{T} \ln \frac{p_1}{p_2}}$$

Meteorologists often refer to "geopotential height" because this quantity is directly associated with energy to vertically displaced air.

$$Geopotential\ Height\ (Z) = \frac{\emptyset}{g_0} = \frac{g\,z}{g_0}$$

g_0 is the globally averaged value of gravity at sea level.

For practical purposes, Z and z are about the same in the troposphere