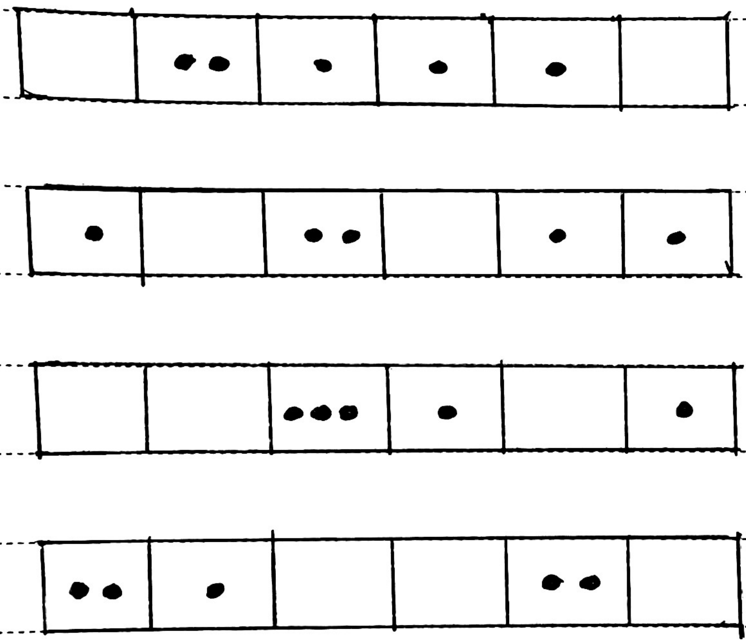
The Bose - Einstein Distribution :-

The configuration of the assembly is specified such that the sheet (S) consist of all the (g_s) states with energies in the range ϵ_s to $\epsilon_s + d\epsilon_s$ contains (n_s) systems. The limitations which are imposed on the allowed values of the occupation numbers (n_s) again arise from the fixed energy (E) and fixed number of systems (N) of the assembly so that:

$$\sum_s n_s \epsilon_s = E \quad \text{and} \quad \sum_s n_s = N \quad \dots (1)$$

Because the systems are indistinguishable the interchang of two systems, whether between two sheets or new arrangement. the only different arrangements which can be formed arise from the arrangement of the systems in the sheets among the states available in the sheets.

Consider the (g_s) states of sheet (S) as represented by the figure below which explain six states and five systems, the dots represent the systems. the (n_s) systems are to be arranged among the (g_s) states.



Arrangement 1

Arrangement 2

Arrangement 3

Arrangement 4

the first state may be chosen in (g_s) ways. the first boundary of this first state may either have one of the (n_s) systems or one of the remaining $(g_s - 1)$ states on its right and this system or state may be chosen in a total of $[(g_s - 1) + n_s]$ ways.

The total number of ways of positioning these (n_s) systems and ($g_s - 1$) states after the first state has been chosen is $[(g_s - 1) + n_s]!$. together with (g_s) ways of selecting the first state, this gives $g_s[(g_s - 1) + n_s]!$ ways of arranging the (n_s) system among the (g_s) states. this number includes ($g_s!$) arrangments of the states among themselve and the ($n_s!$) arrangements of the indistinguishable systems. the number of distinguishable arrangements, (W_s) of the systems whithin the sheet (s) is therefore :

$$W_s = \frac{g_s[(g_s - 1) + n_s]!}{g_s!\, n_s!}$$

$$\therefore W_s = \frac{[(g_s - 1) + n_s]!}{(g_s - 1)!\, n_s!} \quad \text{------} \quad (2)$$

the total number of the arrangements for agiven configuration of the assembly :

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$$W = \prod W_s$$

$$= \prod_s \frac{[(g_s - 1) + n_s]!}{(g_s - 1)!\, n_s!} \quad \text{---} (3)$$

As in the Maxwell-Boltzman statistics the most probable configuration is determined by finding the values of (n_s) which give a maximum value for the weight (W).

$$\sum_s \left(\frac{\partial \log W}{\partial n_s} + \alpha + \beta \epsilon_s \right) dn_s = 0 \quad \text{---} (4)$$

$$\frac{\partial \log W}{\partial n_s} + \alpha + \beta \epsilon_s = 0 \quad \text{---} (5)$$

$$\log W = \sum_s \log W_s$$

$$= \sum_s \log [(g_s - 1) + n_s]! - \log [(g_s - 1)!\, n_s!]$$

By using Stirling's formula:

$$= \sum_s [(g_s - 1 + n_s) \log (g_s - 1 + n_s) - (g_s - 1) \log (g_s - 1) - n_s \log n_s]$$

$$\text{---} (6)$$

$$\therefore \frac{\partial \log W}{\partial n_s} = \log(g_s - 1 + n_s) - \log n_s$$

It has been assumed that both (g_s) & (n_s) are very much greater than unity,

$$\therefore \frac{\partial \log W}{\partial n_s} = \log\left(\frac{g_s + n_s}{n_s}\right) \quad \text{--- } (7)$$

Substituting (7) in (5)

$$\therefore \log\left(\frac{g_s + n_s}{n_s}\right) + \alpha + \beta \epsilon_s = 0$$

$$\therefore \log\left(\frac{g_s}{n_s} + 1\right) = -\alpha - \beta \epsilon_s \Rightarrow \frac{g_s}{n_s} = \left(e^{-\alpha - \beta \epsilon_s} - 1\right)$$

$$\therefore \boxed{n_s = \frac{g_s}{e^{-(\alpha + \beta \epsilon_s)} - 1}}$$

Let $e^{\alpha} = A \Rightarrow e^{-\alpha} = \frac{1}{A}$, $\beta = \frac{-1}{KT}$

$$\therefore \boxed{n_s = \frac{g_s}{\frac{1}{A} e^{e_s/KT} - 1}}$$

Example:

Consider there are two cells i and j and 4 systems distributed over state $g_s = 4$, find the weight configuration (thermodynamics probability)

Sol:

For this case $n_i = 3$, $n_j = 1$

i	4	3	2	1	0
j	0	1	2	3	4

$$\because W_i = \frac{(g_s - 1 + n_i)!}{(g_s - 1)!\, n_i!}$$

$$= \frac{(4-1+3)!}{(4-1)!\, 3!} = \frac{6!}{3!\,3!} = 20$$

$$\because W_j = \frac{(g_s - 1 + n_j)!}{(g_s - 1)!\, n_j!}$$

$$= \frac{(4-1+1)!}{(4-1)!\, 1!} = \frac{4!}{3!\, 1!} = 4$$

$$\therefore W = \prod W_s = 20 \times 40 = 80$$

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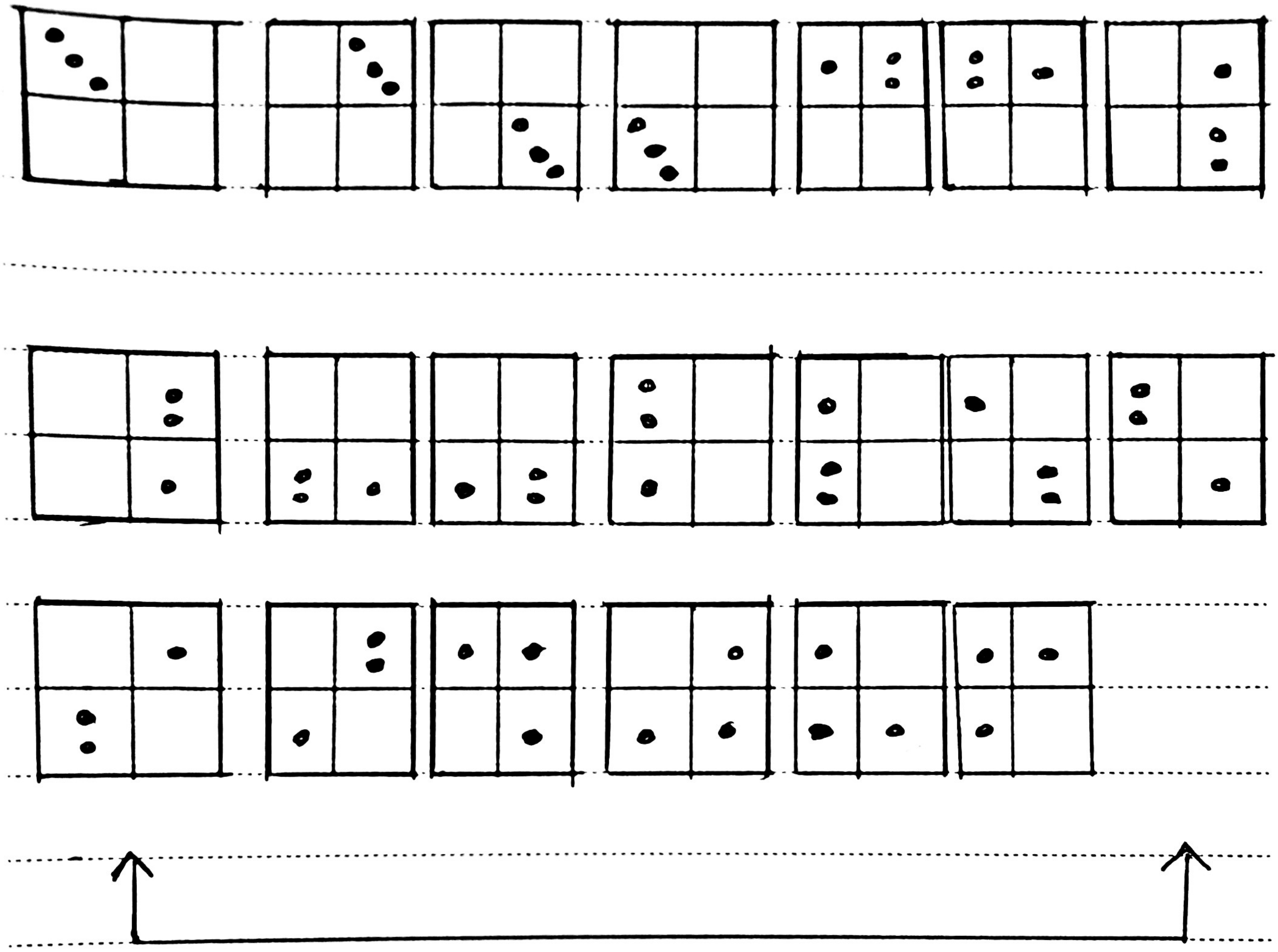
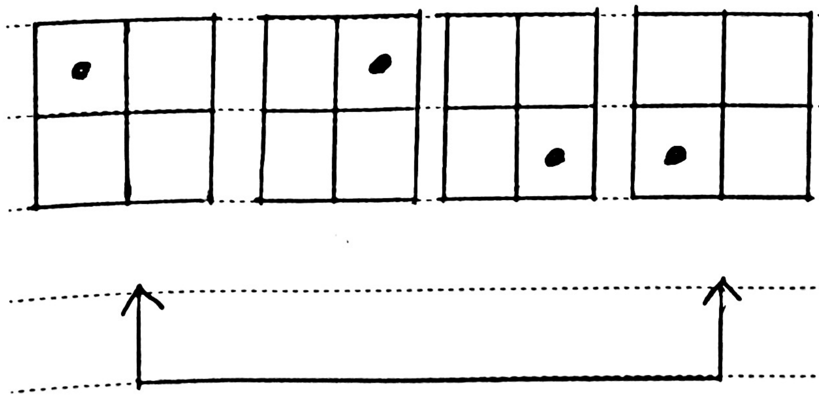


For i cell



For j cell

The occupation number for Bose-Einstein distribution in term of Partition function (Z) is :

$$\boxed{n_s = \frac{N}{Z} \frac{1}{(e^{E_s/K_BT} - 1)}}$$

Example :-

Calculate the number of possible ways to configurate (8) undistinguishable balls in (5) undistinguishable boxes.

$$\therefore W_{B-E} = \frac{[(g_s-1)+n_s]!}{(g_s-1)!\,n_s!}$$

$$= \frac{[(5-1)+8]!}{(5-1)!\,8!} = \frac{12!}{4!\,8!}$$

$$= 495$$