1. Liquid drop model

1. The liquid drop model, proposed in 1935 by Bohr, is based on the following assumptions:
   a) The nucleus consists of incompressible matter so that $R \propto A^{1/3}$.
   b) The nuclear force is identical for every nucleon and in particular does not depend on whether it is a neutron or a proton.
   c) The nuclear force saturates.

2. Nucleons interact strongly with their nearest neighbors, just as molecules do in a drop of water. Therefore, one can attempt to describe their properties by the corresponding quantities, i.e. the radius, the density, the surface tension and the volume energy.

1.1. Semi empirical mass formula (Bethe-Weizsacker mass formula)

3. The mass formula is given by

$$M(A,Z) = Zm_p + Nm_n - \frac{B(A,Z)}{c^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

The last term is the "mass deficit" due to the binding energy $B(A,Z)$.

4. An excellent parameterization of the binding energies of nuclei in their ground state was proposed in 1935 by Bethe and Weizsacker, and is given by:

$$B(A,Z) = a_V A - a_S A^{2/3} - a_C \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \delta(A) + T_{sh} \quad \ldots.. \ (2)$$

where the first, second, third, fourth, fifth and sixth terms are the volume, surface, coulomb, asymmetry, pairing, and shell terms, respectively. The parameters $(a_V, a_S, a_C,$ and $a_a)$ as well as the terms ($\delta(A)$, and T_{sh}), which are determined empirically, have the following values:

$$a_V = 15.753\ MeV,\ a_S = 17.804\ MeV,\ a_C = 0.7103\ MeV,\ a_a = 23.69\ MeV,$$

$$T_{sh} = (1 \to 3)\ MeV \qquad \text{and} \qquad \delta(A) = \begin{cases} 33.6A^{-3/4} & for\ N\ and\ Z\ are\ even \\ -33.6A^{-3/4} & for\ N\ and\ Z\ are\ odd \\ 0 & for\ A = N+Z\ is\ odd \end{cases}$$

Q1. How the volume term, $T_V = a_V A$, is explained by the liquid drop model?

Answer: This term describes that the binding energy is mostly proportional to *A* (or *V*), i.e.

$$T_V \propto \frac{4}{3}\pi r^3 \rightarrow \qquad T_V \propto \frac{4}{3}\pi (R_0 A^{1/3})^3 \rightarrow \qquad T_V \propto \frac{4}{3}\pi R_0^3 A \rightarrow \qquad T_V \propto A$$

$\therefore\ T_V = a_V A$, where a_V is the proportionality constant.

We can imagine the nucleus as a drop of nucleons, each nucleon having attractive force from its neighbors due to the short-ranged nature of the nuclear force. The amount of binding energy is proportional to A (or V), i.e., the volume term is proportional to A (or V).

Q2. How the surface term, $T_S = -a_S A^{2/3}$, is explained by the liquid drop model?

Answer: Again we can imagine the nucleus as a drop of nucleons. Internal nucleons have isotropic interactions, where each of these nucleons having zero resultant forces with its neighbors. While nucleons at the surface receive less binding because they have forces coming only from the inside, where each of these nucleons having none zero resultant forces with its neighbors. Thus the surface term (called the surface tension term), which reduces the binding energy of the nucleus, is proportional to the area of surface, i.e.

$$T_S \propto 4\pi r^2 \rightarrow \qquad T_S \propto 4\pi (R_0 A^{1/3})^2 \rightarrow \qquad T_S \propto 4\pi R_0^2 A^{2/3} \rightarrow \qquad T_S \propto A^{2/3}$$

$\therefore\ T_S = -a_S A^{2/3},$ where $-a_S$ is the proportionality constant.

5. Coulomb repulsion term, $T_C = -a_C \dfrac{Z(Z-1)}{A^{1/3}}$, is proportional to $\dfrac{Z^2}{A^{1/3}}$. This term is smaller than the nuclear terms for small values of Z. It favors a neutron excess over protons. This term has the obvious meaning of total coulomb energy among protons (neutrons are electrically neutral!). It is known that there is coulomb potential between any pair of protons (long-ranged force unlike the nuclear binding force).The typical distance among them is the nuclear size, given by $A^{1/3}$. Coulomb interaction is actually a very weak interaction compared to the nuclear force. Coulomb energy term is important for large nuclei, because it grows like Z^2. This tends to prefer smaller Z for a given A.
6. Asymmetry term, $T_a = -a_a \dfrac{(N-Z)^2}{A} = -a_a \dfrac{(A-2Z)^2}{A}$, is more favorable to have an approximately equal number of protons and neutrons (i.e., $Z = N = A/2$), where stable nuclei (especially for light nuclei) require about $Z = N = A/2$. Thus the isobar with $Z = N = A/2$ is more stable than that with $Z \neq N$, which in sequential leads the binding energy of the isobar with $Z = N$ to be larger than that of $Z \neq N$. It is clear that this term goes to zero for $A = 2Z$ or $N=Z$ and its effect is smaller for larger A (while for smaller nuclei the effect of this term is more important).

Q3. Suppose you have the following isobars: ${}^{16}_{8}O_8$ and ${}^{16}_{7}N_9$, which one of these isobars is larger in binding energy?

Answer: $\mathrm{B}({}^{16}_{8}O_8; Z = N = A/2 = 8) > \mathrm{B}({}^{16}_{7}N_9; Z \neq N)$.

Q4. What do we mean by asymmetry energy?

Answer: It is the difference in the nuclear binding energy of a nucleus with $Z \neq N$ and that of the isobar with $Z = N = A/2$.

7. The pairing term, $T_P = \delta(A)$: There is a tendency that nucleons want to be paired. There is a sizable difference in the binding energies between nuclei with all nucleons paired (even-even ones) and those with some nuclei unpaired (even-odd, odd-even, and odd-odd).
8. The shell term, $T_{sh} = (1 \rightarrow 3)$ MeV:

Q5. Why the shell term T_{sh} is considered in semi empirical mass formula?

Answer: It is considered because the nuclear stability (or the nuclear binding energy) is larger when N or Z (or both) approaches a magic number.

1.2. Successes of liquid drop model

a) It can predict the binding energy of many nuclei correctly.
b) It can predict the emission of α – and β – particles in radioactivity.
c) It can explain the process of the nuclear fission.

1.3. Failures of liquid drop model

a) It fails to explain the extra stability in nuclei when *N* or *Z* (or both) approaches a magic number.
b) It fails to predict the observed magnetic moments of many nuclei.
c) It fails to predict the spin of nuclei.
d) It fails to predict the excited states in most nuclei.

2. Fermi gas model

9. In this model, nuclei are considered as a composed of two fermion gases, a neutron gas and a proton gas. The particles do not interact, but they are confined in a sphere which has the dimension of the nucleus. This model defines properties of a system of non-interacting fermions in an infinite potential well.
10. Fermi case model assumes the following:
    a) All fermions occupy the lowest energy states available up to the Fermi energy E_F, and that there are no excitations across the Fermi energy.
    b) Protons and neutrons are independent fermion filling two separate potential wells.
    c) For stable nuclei, there is a common Fermi energy for the protons and neutrons.
    d) If Fermi energy for protons and neutrons are different then the $\beta -$decay transforms one type of nucleons into the other until the common Fermi energy (stability) is reached.

Notes:

a) Fermi momentum (P_F) is related to Fermi energy by $E_F = P_F^2/2m \rightarrow \;\therefore\; P_F = \sqrt{2mE_F}$, where m is the nucleon mass.
b) For nuclear matter density ($\rho = 0.172$ nucleons/fm^3), $P_F = 268$ MeV/*c* and $E_F = 38$ MeV.

3. Shell model (SM)

Q6. Explain in brief the assumption of the SM.

Answer: In the SM, the nucleons interact with each other through the strongly attractive short range interactions and scatter each other repeatedly. This multiple scattering (interaction) is assumed to give rise an average (mean) potential. The average potential is created such that it consumes all of the nucleon-nucleon (N-N) interaction strength. This means that the particles (after giving rise to the average potential) move freely and independently of each other in this average potential.

3.1. Basic assumptions of the SM

a) Nucleons in a nucleus move independently in an average potential produced by the rest of nucleons.
b) Protons and neutrons separately fill levels in the nucleus.
c) Most of the nucleons are paired and a pair of nucleons contributes zero spin and zero magnetic moment. The paired nucleons thus form an inert core.
d) The properties of odd A nuclei are characterized by the unpaired nucleon and odd-odd nuclei by the unpaired proton and neutron

11. The first step in developing the shell model is the choice of the potential. For the purpose of obtaining the required magic numbers, various forms of potential $V(r)$ were employed. Here we shall focus on the harmonic oscillator potential.

3.2. The harmonic oscillator potential

12. If a particle moves freely and independently in three dimensional harmonic oscillator potential, the Hamiltonian is given by: $H = \frac{p^2}{2m} + \frac{1}{2}m\left(x^2w_x^2 + y^2w_y^2 + z^2w_z^2\right)$.......... .... (3)

    where w_x, w_y and w_z are the angular frequencies in x, y and z directions, respectively.

    For an isotropic oscillator (i.e., $w_x = w_y = w_z$), Eq. (3) becomes: $H = \frac{p^2}{2m} + \frac{1}{2}mr^2w^2$ .... (4)

    where $V(r) = \frac{1}{2}mr^2w^2$ is the harmonic oscillator potential.

13. The single-particle Shrodinger equation (S.E) for a nucleon moves in a harmonic oscillator potential is given by: $H_0\,\phi_{nlm}(\vec{r}) = E_0\,\phi_{nlm}(\vec{r})$ .... (5)

    where the single-particle harmonic oscillator wave function is $\phi_{nlm}(\vec{r}) = R_{nl}(r)Y_{lm}(\theta,\phi)$ ....(6)

    and the symbols $R_{nl}(r)$ and $Y_{lm}(\theta,\phi)$ are the radial and angular parts of $\phi_{nlm}(\vec{r})$, respectively.
14. The sequence of energy levels obtained by solving the S.E. for a particle in a harmonic oscillator potential is shown by Fig. 1.

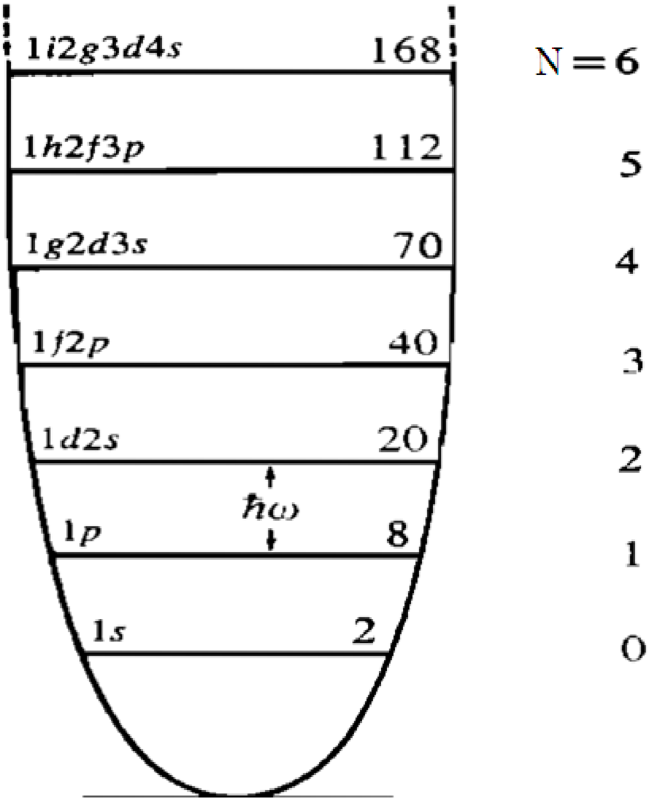

Fig. 1: The single-particle energies of a harmonic oscillator potential as a function of oscillator quantum number N

15. The number of identical nucleons (protons or neutrons) that can occupy each orbital (state) is given by $N_j = 2j+1$

16. The single-particle energy of a particle moves in a harmonic oscillator state $|nl\rangle$, is given by:

$E_0 = (2n+l-1/2)\hbar w$.......... ...... (7)

or $E_0 = (N+3/2)\hbar w$ .... (8)

where N is the oscillator quantum number given by: $N = 2(n-1)+l$ ………………(9)

2.2 *Single-particle states*

APPROXIMATE SEQUENCE OF SINGLE-PARTICLE STATES

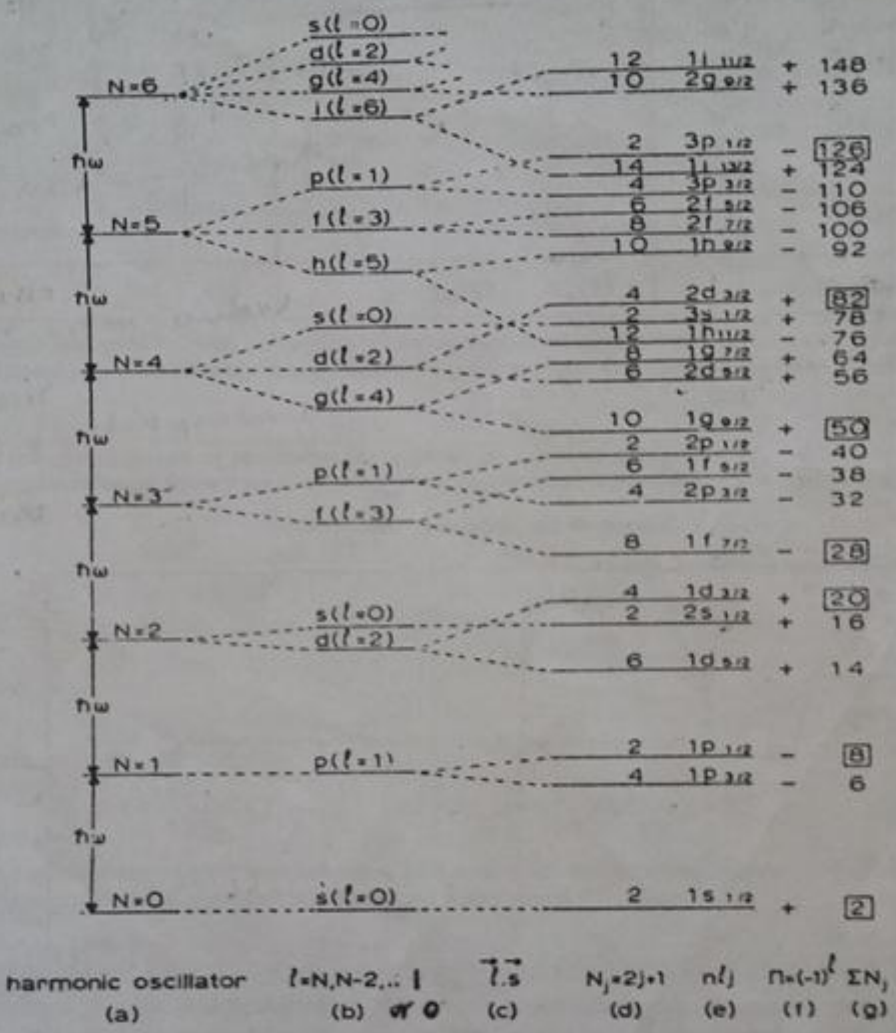

Fig. 2. (a) The single-particle energies of a harmonic-oscillator potential as a function of the oscillator quantum number N. (b) A schematic representation of the single-particle energies of a Saxon-Woods potential. (c) A schematic illustration of the level splitting due to the spin-orbit coupling term. (d) The number $N_j = 2j+1$ of identical particles that can occupy each state. (e) The spectroscopic notation of the single-particle quantum numbers n, l and j. (f) The parity of each state. (g) The magic numbers are seen to appear at the energy gaps as the subtotals of the number of particles.

The level pattern given above represents qualitative features only. This holds especially for states with $N > 4$, where the single-particle level order differs for protons and neutrons and depends also on the number of nucleons occupying lower states.

Table 1: Quantum number for the lower single-particle states of a nucleon in a harmonic oscillator potential

Energy	Shell	N	$\lvert nl\rangle$	n	l	Parity = (-1)^{l}
$3/2\hbar w$	1s	0	1s	1	0	+
$5/2\hbar w$	1p	1	1p	1	1	-
$7/2\hbar w$	1d-2s	2	1d	1	2	+
			2s	2	0	+
$9/2\hbar w$	1f-2p	3	1f	1	3	-
			2p	2	1	-
$11/2\hbar w$	1g-2d-3s	4	1g	1	4	+
			2d	2	2	+
			3s	3	0	

17. The number of particles at shell closure for a pure harmonic oscillator does not agree with the observed magic numbers, especially beyond the 1d-2s shell.
18. The magic number can be reproduced by introducing a strong spin-orbit coupling term into the single particle Hamiltonian.

3.3. The Spin-orbit potential

19. The spin-orbit term is given by: $U_{S.O}(\vec{r}) = f(r)\vec{l}.\vec{s}$ ……………(10)
    where the function $f(r)$ defines the dependence on the radial coordinate (r) and can be related to the central potential (average potential) in which the nucleons move.

    Q12. Derive a formula for the contribution of the spin-orbit term to the energy of the single particle state $\phi_{nljm}(\vec{r})$.

    Answer: The contribution of the spin-orbit term to the energy of the single particle state $\phi_{nljm}(\vec{r})$ is given by the expectation value of the operator $U_{S.O}(\vec{r})$. i.e., $\langle\phi_{nljm}(\vec{r})|U_{S.O}(\vec{r})|\phi_{nljm}(\vec{r})\rangle$ ……………(11)

    Use Eq (10) in Eq. (11) we get

    $\langle\phi_{nljm}(\vec{r})|U_{S.O}(\vec{r})|\phi_{nljm}(\vec{r})\rangle = \langle\phi_{nljm}(\vec{r})|f(r)\vec{l}.\vec{s}|\phi_{nljm}(\vec{r})\rangle$ ……………(12)

    As $\vec{j} = \vec{l} + \vec{s}$ thus we can write $(\vec{j})^2 = (\vec{l} + \vec{s})^2 \rightarrow \therefore\ j^2 = l^2 + s^2 + 2\vec{l}.\vec{s}$ ………. …… (13)

    Eq. (13) can be rewritten as: $\vec{l}.\vec{s} = \frac{1}{2}\left[j^2 - l^2 - s^2\right]$ …………(14)

    Use Eq. (14) in Eq. (12) we get

    $\langle\phi_{nljm}(\vec{r})|U_{S.O}(\vec{r})|\phi_{nljm}(\vec{r})\rangle = \langle\phi_{nljm}(\vec{r})|\frac{1}{2}f(r)\left[j^2 - l^2 - s^2\right]\phi_{nljm}(\vec{r})\rangle$ …………(15)

    The following relations are very useful:

    $j^2|\phi_{nljm}(\vec{r})\rangle = j(j+1)|\phi_{nljm}(\vec{r})\rangle$ …………(16)

    $l^2|\phi_{nljm}(\vec{r})\rangle = l(l+1)|\phi_{nljm}(\vec{r})\rangle$ …………(17)

    $s^2|\phi_{nljm}(\vec{r})\rangle = s(s+1)|\phi_{nljm}(\vec{r})\rangle$ …………(18)

    Use Eqs. (16), (17) and (18) in Eq. (15) we get:

    $\langle\phi_{nljm}(\vec{r})|U_{S.O}(\vec{r})|\phi_{nljm}(\vec{r})\rangle = \langle\phi_{nljm}(\vec{r})|\frac{1}{2}f(r)|\phi_{nljm}(\vec{r})\rangle\ \left[j(j+1) - l(l+1) - s(s+1)\right]$…………(19)

    Use Eq. (6) in Eq. (19) we find

$$\left\langle \phi_{nljm}(\vec{r}) \middle| U_{S.O}(\vec{r}) \middle| \phi_{nljm}(\vec{r}) \right\rangle = \left\langle R_{nl}(r)Y_{lm}(\theta,\phi) \middle| \frac{1}{2} f(r) \middle| R_{nl}(r)Y_{lm}(\theta,\phi) \right\rangle \left[j(j+1) - l(l+1) - s(s+1) \right]$$

$$\rightarrow \qquad = \frac{1}{2} \left\langle R_{nl}(r) \middle| f(r) \middle| R_{nl}(r) \right\rangle \left\langle Y_{lm}(\theta,\phi) \middle| Y_{lm}(\theta,\phi) \right\rangle \left[j(j+1) - l(l+1) - s(s+1) \right]$$

$$\therefore \left\langle \phi_{nljm}(\vec{r}) \middle| U_{S.O}(\vec{r}) \middle| \phi_{nljm}(\vec{r}) \right\rangle = \frac{1}{2} < f(r) >_{nl} \left[j(j+1) - l(l+1) - s(s+1) \right] \ldots\ldots\ldots\ldots(20)$$

where the notation $< f(r) >_{nl} = \langle R_{nl}(r) | f(r) | R_{nl}(r) \rangle \equiv \int_0^\infty R_{nl}(r) R_{nl}(r) f(r) dr$ and the relation $\langle Y_{lm}(\theta,\phi) | Y_{lm}(\theta,\phi) \rangle = 1$ are used in above equation.

Use $j = l \pm s$ together with $s = \frac{1}{2}$, we finally obtain from Eq. (20) the required result.

$$\left\langle \phi_{nljm} \middle| U_{S.O} \middle| \phi_{nljm} \right\rangle \equiv \left\langle U_{S.O} \right\rangle = \begin{cases} -\frac{1}{2}(l+1) < f(r) >_{nl}, & for \quad j = l - \frac{1}{2} \\ \frac{1}{2} l < f(r) >_{nl}, & for \quad j = l + \frac{1}{2} \end{cases} \ldots\ldots\ldots\ldots(21)$$

i.e., Eq. (21) represents the contribution of the spin-orbit term to the energy of the single-particle state $\phi_{nljm}(\vec{r})$.

Q13. Derive a formula for the splitting between $j = l + 1/2$ and $j = l - 1/2$ levels

Answer: The splitting between $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ levels due to the spin-orbit term is given by:

$$\Delta E_{l.s} = \left\langle U_{S.O} \right\rangle_{j=l+1/2} - \left\langle U_{S.O} \right\rangle_{j=l-1/2} \ldots\ldots\ldots\ldots\ldots\ldots(22)$$

Use Eq. (21) in Eq. (22) we get

$$: \Delta E_{l.s} = \frac{1}{2} l < f(r) >_{nl} - \left[-\frac{1}{2}(l+1) < f(r) >_{nl} \right] \quad \rightarrow \quad \therefore \Delta E_{l.s} = < f(r) >_{nl} \frac{2l+1}{2} \ldots\ldots\ldots\ldots \ (23)$$

Notes:

a) It is clear from Eq. (23) that the spin-orbit splitting $\Delta E_{l.s}$ is proportional with the orbital angular momentum (*l*).

b) Empirically, the radial integral $< f(r) >_{nl}$ is approximately given by the relation:

$$< f(r) >_{nl} = -20 A^{-2/3} \text{MeV} \quad \ldots\ldots\ldots\ldots(24)$$

where *A* is the nuclear mass number.

c) The negative value of the radial integral $< f(r) >_{nl}$ reflects the experimental fact that the $j = l + \frac{1}{2}$ level is lowered with respect to the $j = l - \frac{1}{2}$ level.

d) Due to the large splitting of $l = 3$ (f-state), $1f_{7/2}$ level comes somewhere in between $1d_{3/2}$ and $2p_{3/2}$ levels. Therefore $1f_{7/2}$ level gets isolated from the other (fp) states.

e) The occurrence of magic number, which provides the stability of a nucleus, is related to the "energy gap" between two successive levels.

Examples:

a) The $1f_{7/2}$ state gets well separated in energy from other high lying of (fp) states as well as the lower lying of (sd) states. The number of particles filling all the states up to this state ($1f_{7/2}$) is counted as magic number.
b) Similarly, the $1g_{9/2}$ state, due to the $\vec{l}.\vec{s}$ splitting, gets pull down so much that it comes close to the (fp) state energy region. That is why the filling of the (fp) states is not counted as a magic number but when the nucleons complete the filling of $1g_{9/2}$ state then only it provides the magic number.
c) The magic numbers are accounted on the basis of energy gaps between two levels rather than filling up of energy shells.

Q14. For $l = 3$, find the splitting between $j = l + \frac{1}{2}$ and $j = l - \frac{1}{2}$ levels.

Answer: This splitting is given by Eq. (23), i.e., $\Delta E_{l.s} =< f(r) >_{nl} \frac{2l+1}{2}$

Use Eq. (24) together with $l = 3$ in the above equation [Eq. (23)] we obtain: $\Delta E_{l.s} = -20A^{-2/3}\frac{7}{2} = -70A^{-2/3}$

3.4. Successes of the SM

a) It explains the ground state spin and parities of all nuclei.
b) It explains the ground state properties of odd-*A* nuclei.
c) It explains the occurrence of magic numbers which provide the stability of nuclei.
d) It explains the extra stability of magic nuclei.
e) It explains the observed magnetic dipole and electric quadrupole moments of different nuclei.
f) It explains many other properties, such as the nuclear isomerism of different nuclei.

3.5. Failures of the SM

a) It fails to explain the degeneracy of the ground states of even-*A* nuclei. This degeneracy is due to its assumption that the average potential consumes the entire nucleon-nucleon interaction.
b) It fails to describe the observed properties of excited nuclei.

3.6. Extended of the assumption of the SM

20. The assumption of the SM is extended to overcome its failures which are mentioned above in subsection 3.6.

Q10. Explain in brief the extended of the SM.
Answer: It assumes that though the major part of the N-N interaction is consumed in giving rise to an average potential, there is still some residual N-N interaction left. This residual interaction is weak and is responsible for splitting the two nucleon levels. The amount of splitting of levels depends on the strength of the residual interaction.

Q11. How the extended of the SM does overcome the problem of the degeneracy of the ground states of even-*A* nuclei? Support your answer by an example. (As an example, consider ^{18}O nucleus with its outer two neutrons move in the model space of $1d_{5/2}$ and $2s_{1/2}$.

Answer: In ^{18}O nucleus, the 8 protons and 8 neutrons would pair up, and the last two neutrons (according to the SM) would determine the properties of this nucleus. According to the SM, $J^{\pi} = 0^{+}$ is the ground state of ^{18}O while the rest of states will degenerate with this ground state. In the extended of the SM, there is a residual interaction between the two neutrons and this residual interaction will split all these J^{π} states. The nature and magnitude of the splitting depend on the residual interaction (see Fig. 3).

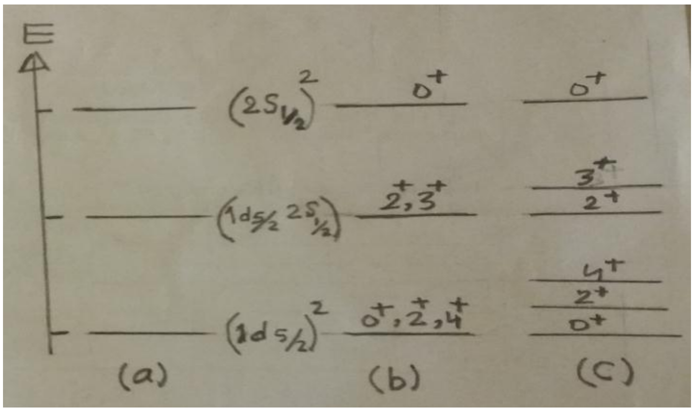

Fig. 3: States of two neutrons in ($1d_{5/2}$ $2s_{1/2}$) configurations:
(a) Unperturbed configurations.
(b) States arising by the SM.
(c) States arising by the extended of SM.

Note: The SM Hamiltonian is given by: $H = H_0 + \sum_{i<j} V_{ij}$ .... (25)

where V_{ij} is only treated as a perturbation and $H_0 = \frac{p^2}{2m_p} + \frac{1}{2} m_p r^2 w^2$. Here $V(r) = \frac{1}{2} m_p r^2 w^2$ is the single-particle harmonic oscillator potential, m_p is the proton mass and $\hbar w$ is the energy quantum of the harmonic oscillator potential.

4. Collective model:

The liquid drop model can account for the behavior of nucleus as a whole, shell model indicates the individual and nearly independent behavior of nucleons. Combining the two aspects together, a new model called collective model was developed. This model gave good results for magnetic moments, excited states, etc. of some nuclei.