Sources of error

مصادر الخطأ ①

The main sources of error in obtaining (نحصل على) numerical Solution to mathematical problems are:

المصدر الرئيسي للخطأ نحصل عليه في الحلول العددية

1- Modeling (تصميم): A mathematical description (وصف) of physical problem usually involves simplifications (التبسيط) and omissions (الإهمال/الحذف)

عادتاً يتضمن الوصف الرياضي لمشكلة فيزيائية التبسيط والحذف

2- Measuring instruments (أدوات القياس) or computing aids (مساعدات حسابية):-

there may be errors in measuring or estimating values

وقد تكون هناك أخطاء في قياس او تقدير القيم

3- The numerical method:

Most of the time numerical methods involve approximations

معظم الوقت يتضمن الطرق العددية تقريباً.

4- Representation of numbers: e.g. π cannot be represented exactly a finite number of digits

* لا يمكن تمثيله بالضبط بعدد محدود من الارقام

5- Arithmetic error (الخطأ الحسابي): Frequently errors are introduced in carrying out operations such as addition (+) and multiplication (x)

يتم ادخال اخطاء في كثير من الاحيان في تنفيذ عمليات مثل الجمع والضرب.

Classification of errors: تصنيف الاخطاء

we classify errors in general into three depending on their sours:

1- Errors which are already present in the statement of a problem before its solution are called

Inherent errors
الأخطاء اعتباطية

يعتبر الاخطاء الموجودة بالفعل في بيانات المسألة قبل حلها بالأخطاء ذاتية.

2- Errors due to (بسبب) arithmetic (الحساب) operation (عمليات) using normalized (الطبيعية) floating-point numbers. Such errors are called rounding errors. (ارقام النقطة العائمة)

Number representation تمثيل الارقام

We assume that machine numbers are represented in the normalized decimal floating-point form.

$\mp 0.d_1 d_2 \cdots d_k \times 10^n$, $1 \leq d_1 \leq 9$ and $0 \leq d_i \leq 9$ for each $i = 2, \ldots, k$.

$1 \leq d$ لا يجوز صفر لأنه

Ex) $x = 284.6054 \longrightarrow x = 0.2846054 \times 10^3$

$y = 0.0324807 \longrightarrow y = 0.324807 \times 10^{-1}$

Number of this form are called k-digit (ارقام) decimal machine (الاعداد العشرية) numbers

Any positive real number within the numerical range of the machine can be normalized to the form

$y = 0.d_1 d_2 d_3 \cdots d_k d_{k+1} d_{k+2} \cdots \times 10^n$

The floating-point form of y, denoted $fl(y)$

There are two ways of performing (تنفيذ) this termination. (هذا الاداء)

1- Chopping القطع

is to simply chop off the digits (ارقام) $d_{k+1} d_{k+2} \cdots$ to obtain $fl(y) = 0.d_1 d_2 \cdots d_k \times 10^n$

Rounding → التدوير (3)

adds $5 \times 10^{n-(k+1)}$ to y, Then chops the result to obtain a number of the form

$$fl(y) = 0.\delta_1 \delta_2 \cdots \delta_k \times 10^n$$

if $d_{k+1} \geq 5$, we add 1 to d_k to obtain $fl(y)$; (round up) & if $d_{k+1} < 5$, we merely chop off all the first k digits; (round down).

Ex) Determinate The five-digit of (π) by using ⓐ chopping ⓑ rounding where $\pi = 0.314159265\ldots \times 10^1$

Solution

ⓐ $fl(\pi) = 0.31415 \times 10^1 = 3.1415$ في حالة الـ chopping تقطع من مراتب بدون ما نضيف للـ 5 واحد منه الى 9

ⓑ $fl(\pi) = 0.3141\underline{6} \times 10^1 = 3.1416$

في حالة الـ rounding اذا الرقم السادس اليمين لخمسة نعطي واحد واذا اقل لا نعطي.

~~Ex) Find the~~

Definition If p^* is an approximation to p.

(1) The absolute error is $|p - p^*|$

(2) The relative error is $\frac{|p - p^*|}{|p|}$; where $p \neq 0$

Ex) Find the absolute error and the relative errors in the following cases-

(a) $p = 0.3000 \times 10^1$ and $p^* = 0.3100 \times 10^1$

(b) $x = 0.3000 \times 10^{-3}$ and $x^* = 0.3100 \times 10^{-3}$

(c) $z = 0.3000 \times 10^4$ and $z^* = 0.3100 \times 10^4$

Solution

(a) $e_{abs} = |p - p^*| = |0.3000 \times 10^1 - 0.31000 \times 10^1| = |0.01000 \times 10^1| = 0.1$

$e_{rel} = \frac{|p - p^*|}{|p|} = \frac{0.1}{|0.3000 \times 10^1|} = 0.3333 \times 10^{-1}$

(b) $e_{abs} = |x - \hat{x}| = |0.3000 \times 10^{-3} - 0.31000 \times 10^{-3}| = 0.1 \times 10^{-4}$

$e_{rel} = \frac{|x - \hat{x}|}{|x|} = \frac{0.1 \times 10^{-4}}{0.3000 \times 10^{-3}} = 0.3333 \times 10^{-1}$

(c) $e_{abs} = |z - z^*| = |0.3000 \times 10^4 - 0.3100 \times 10^4| = 0.1 \times 10^3$

$e_{rel} = \frac{|z - z^*|}{|z|} = \frac{0.1 \times 10^3}{|0.3000 \times 10^4|} = 0.3333 \times 10^{-1}$

Definition The number p^* is said to approximate p to t siginificant (الارقام المعنوية) digits (or figures) if t is the largest nonnegative integer for which

$$\frac{|p - p^*|}{|p|} < 5 \times 10^{-t}$$

The floating-point representation $fl(y)$ for the number y has the relative error $\left| \frac{y - fl(y)}{y} \right|$

If k decimal digits and chopping are used for the machine representation of

$$y = 0.d_1 d_2 \cdots d_k d_{k+1} \cdots \times 10^n$$

Then

الفرق صار 10^{-k} k من

Then

$$\left|\frac{y - fl(y)}{y}\right| = \left|\frac{0.d_1d_2\ldots d_kd_{k+1}\cdots \times 10^n - 0.d_1d_2\ldots d_k \times 10^n}{0.d_1d_2\ldots d_kd_{k+1}\cdots \times 10^n}\right|$$

$$= \left|\frac{0.d_{k+1}d_{k+2}\cdots \times 10^{n-k}}{0.d_1d_2\cdots \times 10^n}\right| \#$$

$$= \left|\frac{0.d_{k+1}d_{k+2}\cdots}{0.d_1d_2\cdots}\right| \times 10^{-k}$$

Since $d_1 \neq 0$, The minimal (الأقل) value of the denominator (المقام) is 0.1. The numerator (البسط) is bounded by 1.

$$\left|\frac{y - fl(y)}{y}\right| \leq \frac{1}{0.1} \times 10^{-k} = 10^{-k+1}$$

H.W

EX) Compute The absolute error and relative error in approximations of:

ⓐ $p = \pi$, $p^* = \frac{22}{7}$

ⓑ $p = \pi$, $p^* = 3.1416$

ⓒ $x = e$, $x^* = 2.718$

ⓓ $x = \sqrt{2}$, $\hat{x} = 1.414$

الأخطاء في العمليات الحسابية

لتكن $\hat{x}$ قيمة تقريبية لقيمة حقيقية للعدد x وبخطأ مطلق e_x وخطأ نسبي δ_x و $\hat{y}$ قيمة تقريبية حقيقية للعدد y وبخطأ مطلق e_y وخطأ نسبي δ_y فإن انتشار الأخطاء على العمليات الحسابية يتم كما يلي:

1- خطأ عملية الجمع

$$e_{x+y} = (x+y)-(\hat{x}+\hat{y})$$
$$= x+y-\hat{x}-\hat{y} = (x-\hat{x})+(y-\hat{y}) = e_x + e_y$$

2- خطأ عملية الطرح

$$e_{x-y} = (x-y)-(\hat{x}-\hat{y})$$
$$= x-y-\hat{x}+\hat{y} = (x-\hat{x})-(y-\hat{y}) = e_x - e_y$$

3- خطأ عملية الضرب

$$e_{xy} = (xy)-(\hat{x}\hat{y})$$
$$= xy-(x-e_x)(y-e_y) = \cancel{xy} - \cancel{xy} + xe_y + ye_x - \underbrace{e_x e_y}_{\text{يهمل}}$$

$$\Rightarrow e_{xy} = xe_y + ye_x$$

4- خطأ عملية القسمة

$$e_{\frac{x}{y}} = \left(\frac{x}{y}\right)-\left(\frac{\hat{x}}{\hat{y}}\right)$$

$$\therefore \left(\frac{\hat{x}}{\hat{y}}\right) = \frac{x-e_x}{y-e_y} = \frac{x-e_x}{y\left(1-\frac{e_y}{y}\right)} = \frac{x-e_x}{y}\left(\frac{1}{1-\frac{e_y}{y}}\right)$$

$$= \left(\frac{x-e_x}{y}\right)*\left(1+\frac{e_y}{y}+\left(\frac{e_y}{y}\right)^2+\cdots\right)$$ ← يهمل لصغر قيمته

$$= \left(\frac{x-e_x}{y}\right)\left(1+\frac{e_y}{y}\right) = \left(\frac{x}{y}-\frac{e_x}{y}\right)\left(1+\frac{e_y}{y}\right)$$

$$= \frac{x}{y}+\frac{xe_y}{y^2}-\frac{e_x}{y}-\underbrace{\frac{e_xe_y}{y^2}}_{\text{يهمل}}$$

$$\therefore e_{\frac{x}{y}} = \frac{x}{y}-\left(\frac{x}{y}+\frac{xe_y}{y^2}-\frac{e_x}{y}\right) = \cancel{\frac{x}{y}}-\cancel{\frac{x}{y}}-\frac{xe_y}{y^2}+\frac{e_x}{y}$$
$$= \frac{e_x}{y}-\frac{xe_y}{y^2}$$

$$e_{\frac{x}{y}} = \frac{x}{y}\left(\frac{e_x}{x} - \frac{e_y}{y}\right)$$

ثانياً الخطأ النسبي

١- خطأ عملية الجمع

$$\delta_{x+y} = \frac{e_{x+y}}{x+y} = \frac{1}{x+y}(e_x + e_y)$$

$$= \frac{e_x}{x+y} + \frac{e_y}{x+y} = \frac{x}{x+y}\frac{e_x}{x} + \frac{y}{x+y}\frac{e_y}{y}$$

$$\boxed{\delta_{x+y} = \frac{x}{x+y}\delta_x + \frac{y}{x+y}\delta_y}$$

٢- خطأ عملية الطرح

$$\boxed{\delta_{x-y} = \frac{x}{x-y}\delta_x - \frac{y}{x+y}\delta_y} \rightarrow H.W$$

٣- خطأ عملية الضرب

$$\delta_{xy} = \frac{e_{xy}}{xy} = \frac{xe_y + ye_x}{xy} = \frac{xe_y}{xy} + \frac{ye_x}{xy}$$

$$= \frac{e_y}{y} + \frac{e_x}{x}$$

$$\boxed{\therefore \delta_{xy} = \delta_x + \delta_y}$$

٤- خطأ عملية القسمة

$$\delta_{\frac{x}{y}} = \frac{e_{\frac{x}{y}}}{\frac{x}{y}} = \frac{\frac{x}{y}\left(\frac{e_x}{x} - \frac{e_y}{y}\right)}{\frac{x}{y}} = \frac{e_x}{x} - \frac{e_y}{y}$$

$$\boxed{\therefore \delta_{\frac{x}{y}} = \delta_x - \delta_y}$$

Ex) Find the absolute and relative errors of

1. $Z = \frac{x^2}{y} + 3y$ 2. (H.W) $Z = \frac{x}{y} - 3x^2y$ 3. (H.W) $Z = \frac{2x+y}{x+1}$

Solution

① $Z = \frac{x^2}{y} + 3y$

$$e_Z = e_{\left(\frac{x^2}{y} + 3y\right)} = e_{\left(\frac{x^2}{y^2}\right)} + e_{3y} = \frac{x^2}{y}\left(\frac{e_{x^2}}{x^2} - \frac{e_y}{y}\right) + 3e_y + y e_3$$

($e_{x^2} \rightarrow e_{x \cdot x}$, e_3: صفر)

$$= \frac{x^2}{y}\left[\left(\frac{x e_x + x e_x}{x^2}\right) - \frac{e_y}{y}\right] + 3e_y = \frac{x^2}{y}\left(\frac{2x e_x}{x^2} - \frac{e_y}{y}\right) + 3e_y$$

$$= \frac{x^2}{y}\left(\frac{2e_x}{x} - \frac{e_y}{y}\right) + 3e_y$$

$$\delta_z = \delta_{\left(\frac{x^2}{y} + 3y\right)} = \frac{\frac{x^2}{y}}{z}\,\delta_{\frac{x^2}{y}} + \frac{3y}{z}\,\delta_{3y}$$

$$= \frac{x^2/y}{z}\left[\delta_{x^2} - \delta_y\right] + \frac{3y}{z}\left[\delta_3 + \delta_y\right]$$

$$= \frac{x^2/y}{z}\left[2\delta_x - \delta_y\right] + \frac{3y}{z}\delta_y$$

<u>Example</u> write the following numbers to the floating-point formula

$X = 25149 \longrightarrow fl(x) = 0.25149 * 10^5$

$y = 0.00398 \longrightarrow fl(y) = 0.398 * 10^{-2}$

$Z = 0.733 \longrightarrow fl(Z) = 0.733 * 10^0$

العمليات الحسابية للاعداد بصيغة الفارزة السائبة (floating point)

١- عمليتا الجمع والطرح

لكي $y_1 = A_1 * B_1^{c_1}$ و $y = A_2 * B_2^{c_2}$ تجمع او تطرح صنف العددين

يجب ان يتوفر شرطين

① $B_1 = B_2$ & ② $c_1 = c_2$

② عمليتا الضرب والقسمة

عند ضرب أو قسمة العددين اعلاه فانه يجب ان يتوفر الشرط الاتي فقط $B_1 = B_2$

Example If $x = 22.159$, $y = 0.03$ & $z = 111$, then the floating point formula find

1. $K = 2x + \frac{y+z}{x}$ 2. $K = \frac{x^2+1}{y} - yz$ 3. $K = x - 2y + x$

Solution $x = 22.159 \longrightarrow fl(x) = 0.22159 \times 10^{2}$

$y = 0.03 \longrightarrow fl(y) = 0.3 \times 10^{-1}$

$z = 111 \longrightarrow fl(z) = 0.111 \times 10^{3}$

① $K = 2x + \frac{y+z}{x}$

$2x = 2(0.2 \times 10^{1}) * (0.22159 \times 10^{2}) = 0.044318 * 10^{3}$

$= 0.44318 \times 10^{-1} * 10^{3}$

$y + z = (0.3 * 10^{-1}) + (0.111 * 10^{3}) = 0.00003 * 10^{3} + 0.111 * 10^{3}$

$= 0.11103 * 10^{3}$

$\frac{y+z}{x} = \frac{0.11103 * 10^{3}}{0.22159 * 10^{2}} = 0.50106051 7 * 10^{3} * 10^{-2}$

$= 0.501060517 * 10^{1}$

$\therefore K = 2x + \frac{y+z}{x} = 0.44318 * 10^{2} + 0.501060517 * 10^{1}$

$= 0.44318 * 10^{2} + 0.0501060517 * 10^{2}$

$\therefore K = 0.4932860517 * 10^{2}$

② H.W and ③ H.W