1.3.12. Algebra of Logical Proposition

The logical equivalences below are important equivalences that should be memorized.

1-Identity Laws: $p \wedge T \equiv p$.
$p \vee F \equiv p$.

2-Domination Laws: $p \vee T \equiv T$.
$p \wedge F \equiv F$.

3-Idempotent Laws: $p \vee p \equiv p$.
$p \wedge p \equiv p$.

4- Double Negation Law: $\sim(\sim p) \equiv p$.

5- Commutative Laws: $p \vee q \equiv q \vee p$.
$p \wedge q \equiv q \wedge p$.

6- Associative Laws: $(p \vee q) \vee r \equiv p \vee (q \vee r)$.
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

7- Distributive Laws: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

8- De Morgan's Laws: $\sim (p \wedge q) \equiv \sim p \vee \sim q$.
$\sim (p \vee q) \equiv \sim p \wedge \sim q$.

9- Absorption Laws: $p \wedge (p \vee q) \equiv p$.
$p \vee (p \wedge q) \equiv p$.
$p \wedge (\sim p \vee q) \equiv p \wedge q$.
$p \vee (\sim p \wedge q) \equiv p \vee q$.

10-Implication Law: $(p \rightarrow q) \equiv (\sim p \vee q)$.

11- Contrapositive Law: $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$.

12- Tautology: $p \vee \sim p \equiv T$.

13- Contradiction: $p \wedge \sim p \equiv F$.

14- Equivalence: $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

15- $p \veebar q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

Solution.

(8) We using truth table to prove $\sim (p \wedge q) \equiv \sim p \vee \sim q$.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(14) We using truth table to prove $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (p \leftrightarrow q)$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \rightarrow q \wedge q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

(15) $p \veebar q \equiv (p \vee q) \wedge \sim(p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$\mathbf{p \veebar q}$	$(p \vee q) \wedge \sim (p \wedge q)$
T	T	T	T	F	F	F
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	F	T	F	F

1.4. Rules of Proof

1.4.1.

(i) Rule of Replacement.

Any term in a logical formula may be replaced be an equivalent term.

For instance, if $q \equiv r$, then $p \wedge q \equiv p \wedge r$ Rep(q:r).

(ii) Rule of Substitution.

A sentence which is obtained by substituting logical propositions for the terms of a theorem is itself a theorem.

For instance, $(p \rightarrow q) \vee w \equiv w \vee (p \rightarrow q)$ Sub(p: $p \rightarrow q$), in Commutative Law $p \vee w \equiv w \vee p$.

(iii) Rule of Inference.

1-	$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$	6-	$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore p \rightarrow r \end{array}$
2-	$\begin{array}{l} \sim q \\ \underline{p \rightarrow q} \\ \therefore \sim p \end{array}$	7-	$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore q \end{array}$
3-	$\begin{array}{l} p \\ \hline \therefore p \vee R \end{array}$	8-	$\begin{array}{l} p \vee q \\ \underline{\sim p \vee r} \\ \therefore q \vee r \end{array}$
4-	$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$	9-	$\begin{array}{l} p \rightarrow q \\ \underline{r \rightarrow t} \\ \therefore p \vee r \rightarrow q \vee t \end{array}$

5-	$\frac{p \wedge q}{\therefore p}$	10-	$\frac{\begin{array}{c} p \\ q \rightarrow r \end{array}}{\therefore p \vee q \rightarrow p \vee r}$

Example 1.4.2.

(i) Given
(1) $p \wedge q$

(2) $p \rightarrow \sim (q \wedge r)$

(3) $s \rightarrow r$

$\therefore \sim s$

Solution:

1- $p \wedge q$	1st hypothesis(premise)
2- p	Inf. (1) Properties of $\wedge$
3- q	Inf. (1) Properties of $\wedge$
4- $p \rightarrow \sim (q \wedge r)$	2nd hypothesis(premise)
5- $\sim (q \wedge r)$	Inf. (2),(4)
6- $\sim q \vee \sim r$	De Morgan's Law on (5)
7- $\sim r$	Inf. (3),(6) and Domination Laws
8- $s \rightarrow r$	3rd hypothesis(premise)
9- $\sim r \rightarrow \sim s$	Contrapositive Law
10- $\sim s$	Inf. (7),(9)

(ii) Given

(1) $\sim (p \vee q) \rightarrow r$

(2) $\sim p$

(3) $\sim r$

$\therefore q$

Solution:

1- $\sim (p \vee q) \rightarrow r$ 1st hypothesis(premise)

2- $\sim r$ 3rd hypothesis(premise)

3- $\sim r \rightarrow (p \vee q)$ Contrapositive Law and Double Negation Law

4- $p \vee q$ Inf. (2),(3)

5- $\sim p$ 2nd hypothesis(premise)

6- q Inf. (4),(5)

(iii) Given

(1) $\sim p \rightarrow (r \wedge s)$

(2) $p \rightarrow q$

(3) $\sim q$

$\therefore r$

Solution:

1- $p \rightarrow q$ 2nd hypothesis(premise)

2- $\sim q \rightarrow \sim p$ Contrapositive Law on (1)

3- $\sim q$ 3rd hypothesis(premise)

4- $\sim p$ Inf. (2),(3)

5- $\sim p \rightarrow (r \wedge s)$ 1st hypothesis(premise)

6- $r \wedge s$ Inf. (4),(5)

7- r Inf. (6) Properties of $\wedge$

(iv) Given

(1) $p \rightarrow (\sim r \wedge \sim s)$

(2) $p \vee \sim q$

(3) s

$\therefore \sim q \wedge s$

Solution:

1- $p \rightarrow (\sim r \wedge \sim s)$ 1st hypothesis(premise)

2- $(r \vee s) \rightarrow \sim p$ Contrapositive Law on (1)

3- $p \vee \sim q$ 2nd hypothesis(premise)

4- $\sim p \rightarrow \sim q$ Implication Law on (3)

5-$(r \vee s) \rightarrow \sim q$ Inf. (2),(4)

6- s 3rd hypothesis(premise)

7-$r \vee s$ Inf. (6)

8- $\sim q$ Inf. (5),(7)

9- $\sim q \wedge s$ Inf. (6),(8)

(v) Given

(1) $p \vee q$

(2) $q \rightarrow r$

(3) $\sim r$

$\therefore p$

Solution:

1- $q \rightarrow r$	2^{nd} hypothesis(premise)
2- $\sim r \rightarrow \sim q$	Contrapositive Law on (1)
3- $\sim r$	3^{rd} hypothesis(premise)
4- $\sim q$	Inf. (2),(3)
5- $p \vee q$	1^{st} hypothesis(premise)
6- $(p \vee q) \wedge \sim q$	Inf. (4),(5)
7- $(p \wedge \sim q) \vee (q \wedge \sim q)$	Distributive Law on (6)
8- $(p \wedge \sim q) \vee F$	Contradiction Law (7)
9- $(p \wedge \sim q)$	Identity Law on (8)
10- p	Inf. (9) properties of $\wedge$

(vi) Given
(1) "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"
(2) "If the sailing race is held, then the cup will be awarded"
(3) "The cup was not awarded"
Does this imply that: "It rained"?
Solution.
p: rain
q: foggy
r: the sailing race will be held
s: the lifesaving demonstration will go on
t: then the cup will be awarded

Symbolically, the proposition is

$(1) \sim p \vee \sim q \rightarrow r \wedge s$
$(2) \quad r \rightarrow t$
$(3) \quad \sim t$
$\therefore \quad p$

1. $\sim t$	3rd hypothesis
2. $r \rightarrow t$	2nd hypothesis
3. $\sim t \rightarrow \sim r$	Contrapositive of 2
4. $\sim r$	Inf. (1),(3)
5. $\sim p \vee \sim q \rightarrow r \wedge s$	1st hypothesis
6. $\sim(r \wedge s) \rightarrow \sim(\sim p \vee \sim q)$	Contrapositive of 5
7. $\sim r \vee \sim s \rightarrow (p \wedge q)$	De Morgan's law and double negation law on (5)
8. $\sim r \vee \sim s$	Inf. (4) and domination law
9. $p \wedge q$	Inf. (7),(8)
10. p	Inf. (9)

Example 1.4.3. Use the logical equivalences to show that
(i) $\sim(p \rightarrow q) \equiv p \wedge \sim q$,
(ii) $\sim(p \vee \sim(p \wedge q))$ is a contradiction,
(iii) $\sim (p \vee (\sim p \wedge q)) \equiv (\sim p \wedge \sim q)$,

(iv) $p \vee (p \wedge q) \equiv p$ (Absorption Law).

Solution.

(i) $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q)$ Implication Law

$\equiv \sim(\sim p) \wedge \sim q$. De Morgan's Law
$\equiv p \wedge \sim q$ Double Negation Law

(ii) $\sim(p \vee \sim(p \wedge q))$

$\equiv \sim p \wedge \sim(\sim(p \wedge q))$ De Morgan's Law
$\equiv \sim p \wedge (p \wedge q)$ Double Negation Law
$\equiv (\sim p \wedge p) \wedge q$ Associative Law
$\equiv F \wedge q$ Contradiction Law
$\equiv F$ Domination Law and Commutative Law.

(iii) $\sim (p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim (\sim p \wedge q)$ De Morgan's Law

$\equiv \sim p \wedge (\sim\sim p \vee \sim q)$ De Morgan's Law

$\equiv \sim p \wedge (p \vee \sim q)$ Double Negation Law

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ Distribution Law

$\equiv (p \wedge \sim p) \vee (\sim p \wedge \sim q)$ Commutative Law

$\equiv F \vee (\sim p \wedge \sim q)$ Contradiction Law

$\equiv (\sim p \wedge \sim q) \vee F$ Commutative Law

$\equiv (\sim p \wedge \sim q)$ Identity Law

(iv) $p \vee (p \wedge q)$

$\equiv (p \wedge T) \vee (p \wedge q)$ Identity Law (in reverse)
$\equiv p \wedge (T \vee q)$ Distributive Law (in reverse)
$\equiv p \wedge T$ Domination Law
$\equiv p$ Identity Law

Example 1.4.4. Find a simple form for the negation of the proposition
"If the sun is shining, then I am going to the ball game."
Solution.

p: the sun is shining q: I am going to the football game

This proposition is of the form $p \rightarrow q$. Since $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv (p \wedge \sim q)$.This is the proposition **"The sun is shining, and I am not going to the football game."**