1.5. Logical Implication

Definition 1.5.1. (Logical implication)

We say the logical proposition **"r"** implies the logical proposition **"s"** (or **s** logically deduced from **r**) and write (**r** ⟹ **s**) iff (**r** → **s**) is a tautology.

Example 1.5.2. Show that [(p → t) ∧ (t →q)] ⟹ (p → q).

Solution. Let P: the proposition (p → t) ∧ (t →q)

Q: the proposition p → q

p	t	q	p → t	t → **q**	P	Q	P → Q
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Remark 1.5.3.

(i) We use (r ⟹ s) to imply that the statement (r → s) is true, while the statement (r → s) alone does not imply any particular truth value. The symbol is often used in proofs as shorthand for "**implies**".

(ii) If (r ⟹ s) and (s ⟹ r), then we denote that by (r ⟺s).

Example 1.5.4. Show that
(i) (r ⟹ s) ≡ [(~r ∨ s) is tautology].

(ii) $(r \Leftrightarrow s) \equiv (r \equiv s)$.

Solution.
(i)
1- $(r \Longrightarrow s) \equiv (r \to s)$ is tautology (Def. of $\Longrightarrow$)
2- $(r \to s) \equiv (\sim r \vee s)$ Logical Implication Law
3- $(\sim r \vee s)$ is tautology Inf. (1),(2)

(ii)
1- $\left.\begin{array}{l}(r \Longrightarrow s) \equiv (r \to s) \text{ is tautology} \\ (s \Longrightarrow r) \equiv (s \to r) \text{ is tautology}\end{array}\right\}$ Def. of $(\Longrightarrow)$ and $(\Leftrightarrow)$

2- $(r \to s) \wedge (s \to r)$ is tautology.
3- $r \leftrightarrow s$ is tautology Equivalence Law
4- $r \equiv s$. Inf. Remark 1.3.11

- Generally, the statement and its converse not necessary equivalent. Therefore, $p \Longrightarrow q$ does not mean that $q \Longrightarrow p$.

Example 1.5.5. The statement “the triangle which has equal sides, has two equal legs” equivalent to the statement “ the triangle which has not two equal legs has no equal sides”.

1.6. Quantifiers

Definition 1.6.1.
(i) A **predicate** or **propositional function** is a statement (formula) containing variables and that may be true or false depending on the values of these variables.

- That is, a predicate is a property or relationship between objects represented symbolically.
- We represent a predicate by a letter followed by the variables enclosed between parenthesis: $P(x)$, $Q(x, y)$, etc.

(ii) An ***example*** for $P(x)$ is: value of x for which $P(x)$ is true.
(iii) A ***counterexample*** $P(x)$ is: value of x for which $P(x)$ is false.
(iv) The set, X which contain all possible value that satisfy the formula P is called a **universal set**.
(v) A set Y which contains all values x belong to set X such that $P(x)$ is true is called a **solution set.**

$$Y = S_P = \{x \in X : P(x) \text{is true}\}$$

Example 1.6.2.

(i) $P(x) = x \leq 5 \wedge x > 3$ is true for $x = 4$ and false for $x = 6$ (counterexample).
(ii) $P(x) = x \leq 5 \wedge x > 3$, for every real numbers, x which is definitely false.

(iii) There exists an x such that $P(x) = x \leq 5 \wedge x > 3$," which is definitely true.

(iv) Given the statement "**Ahmad is a logician**".

Let P represent **'is a logician'** and let x represent **'Ahmad'**. The predicate form of this statement is $P(x)$. That is, $P(x) =$ Ahmad is a logician.

(v) Let r: x is married to y.

Let M represent **"married".** Then, $\mathrm{r} = M(x, y)$.

(vi) Let r: The numbers x and y are both odd.
This statement means (x is odd) ∧ (y is odd).

Let P represent **'is a odd'** and let x, y represent **'numbers'**. The predicate form of this statement is $P(x) \wedge P(y)$.

Definition 1.6.3.

(i) The phrase **"for all x"** (**"for every x"**, **"for each x"**) is called a **universal quantifier** and is denoted by $\forall\, x$.

(ii) The phrase **"for some x"** (**"there exists an x"**) is called an **existential quantifier** and is denoted by $\exists\, x$.

Definition 1.6.4. (The Universal Quantifier Proposition)

Let $f(x)$ be a proposition function which depend only on x. A sentence $\forall\, x, f(x)$ read "For all x, $P(x)$" mean

"For all values x in X(universal set), the predicate $f(x)$ is true."; that is,

$$\frac{\forall x, f(x)}{\therefore\ f(a)}$$

Example 1.6.5.

(i) r: The square of all real numbers are positive.

$$\boxed{\text{r}: \forall x \in \mathbb{R}, \quad (x^{2} \geq 0).}$$

(ii) r: The commutative law of addition of real numbers is holed.

$$\boxed{\text{r}: \forall\, x, \forall\, y \in \mathbb{R}, \quad (x + y = y + x).}$$

(iii) r: The associative law of addition of real numbers is holed.

$$\boxed{\text{r}: \forall\, x, \forall\, y, \forall\, z \in \mathbb{R}, \ \big((x + y) + z = x + (y + z)\big).}$$

(iv) r: All logicians are exceptional.
Let L represent **'set of logician'** and let E represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r}: \forall x \in L, \ E(x)}$.

(v) r: All cars are red.
Let X ≔ **Set of cars**, f:= **is red**. Then, r : $\forall x \in X, f(x)$.

Remark .1.6.6.

(i) The "**all**" form, the universal quantifier, is frequently encountered in the following context: $\forall x\ (f(x) \longrightarrow Q(x))$,

which may be read,

"For all x in a universal set X satisfying $f(x)$ also satisfy $Q(x)$".

For example:

(a) r: All logicians are exceptional.

Let L represent **'is a logician'** and let E represent **'is exceptional'**. Then

- Predicate Logic: r: $\forall x(L(x) \longrightarrow E(x))$
- In logical English: "For all x, if x is a logician, then x is exceptional."

(b) r: The square of all real numbers are positive.
Let P represent: $\in \mathbb{R}$ and let Q represent **" is positive"**.
- Predicate Logic: r: $\forall x(P(x) \longrightarrow Q(x))$; that is,

$$\boxed{\text{r}: \forall x(\text{ if } x \in \mathbb{R} \longrightarrow (x^{2} \geq 0).}$$

- In logical English: "For all x, if x is real number, then x is positive."

(c) Every(each, any) integer number is even (or: Integer numbers are even).

Let P represent: $\in \mathbb{Z}$ and let E represent **" is even"**.
- Predicate Logic: r: $\forall x(P(x) \longrightarrow E(x))$; that is,

$$\boxed{\text{r}: \forall x(\text{ if } x \in \mathbb{Z} \longrightarrow E(x)).}$$

- In logical English: "For all x, if x is an integer, then x is even."

(ii) Parentheses are crucial here; be sure you understand the difference between the "all" form and $\boxed{\forall x, f(x) \longrightarrow \forall x, Q(x)}$ and $\boxed{(\forall x,\ f(x)) \longrightarrow Q(x)}$.

Definition 1.6.7. (The Existential Quantifier Proposition)

A sentence $\exists x, f(x)$ read "For some x, $P(x)$" or "For some x such that $P(x)$" mean

"For some $x \in X$(universal set), the predicate $f(x)$ is true"; that is,

$$\frac{f(a)}{\therefore \exists x, f(x)}$$

Example 1.6.8.

(i) $\exists\, x: (x \geq x^2)$ is true since $x = 0$ is a solution. There are many others.

(ii) r: Some logicians are exceptional.
Let L represent **'set of logician'** and let E represent **'is exceptional'**. The predicate form of this statement is $\boxed{\text{r: } \exists x \in L,\ E(x)}$.

(iii) r: There is a car which is red.
Let $X :=$ **Set of cars**, $f :=$ **is red**. Then, r : $\exists x \in X, f(x)$.

Remark .1.6.9.

(i) The "**some**" form, the existential quantifier, is frequently encountered in the following context: $\exists x\ (f(x) \wedge Q(x))$,

which may be read,

"Some x in a universal set X satisfying $f(x)$ and satisfy $Q(x)$".

For example:

(a) r: Some logicians are exceptional.

Let L represent **'is a logician'** and let E represent **'is exceptional'**. Then

- Predicate Logic: r: $\exists x(L(x) \wedge E(x))$
- In logical English: "For some x, x is a logician and x is exceptional."

(b) r: The square of some integers numbers are four (or: There is an integer for which its square is four)
Let P represent: $\in \mathbb{Z}$ and let Q represent **" is 4"**.

- Predicate Logic: r: $\exists x(P(x) \wedge Q(x))$; that is,

$$\boxed{\text{r: } \exists x(\, x \in \mathbb{Z} \ \wedge\ x^2 = 4).}$$

- In logical English: "For some x, x is an integer number and $x^2 = 4$."

(c) At least one integer number is even (or: Some Integers are even).

Let P represent: $\in \mathbb{Z}$ and let E represent **" is even"**.

- Predicate Logic: r: $\exists x(P(x) \wedge E(x))$; that is,

  $$\boxed{\text{r: } \exists x(x \in \mathbb{Z} \wedge E(x)).}$$
- In logical English: "For some x, x is an integer number and x is even."

Negation Rules of Quantifiers 1.6.10.
(i) When we negate a quantified statement, we negate all the quantifiers first, from left to right (keeping the same order), then we negative the statement.
(ii) $\sim (x = y) = (x \neq y)$.
(iii) $\sim (x \equiv y) = (x \not\equiv y)$.
(iv) $\sim (x < y) = (y \leq x)$.
(v) $\sim (x \in Y) = (x \notin Y)$.
(vi) ~ (Even number) = Odd number.

Now define the a formal universal quantifier proposition using negation.

Definition 1.6.11.

(i) $\forall x, f(x) = \sim\exists\, x, \sim f(x)$.
(ii) $\exists\, x, f(x) \equiv \sim\forall x, \sim f(x)$.

Example 1.6.12.
r: All logicians are exceptional.

Let L represent **'set of logician'** and let E represent **'is exceptional'**.

- Predicate Logic: $\boxed{\text{r: } \forall x \in L,\ \ E(x) = \sim\exists\, x, \sim E(x)}$.
- In logical English: "There is no x is a logician, for which x is not exceptional."

Equivalent Definitions 1.6.13.
(i) $\sim\big(\forall x, f(x)\big) \equiv\ \exists\, x, \sim f(x)$.
(ii) $\sim\big(\exists\, x, f(x)\big) \equiv \forall\, x, \sim f(x)$.
(iii) $\sim\ \big[\forall x\ \big(f(x) \longrightarrow\ Q(x)\big)\big] \equiv \exists x\ \big(f(x) \wedge\ \sim Q(x)\big)$
$\equiv$ Some $f(x)$ are not $Q(x)$
(iv) $\sim \Big(\exists\, x, \big(f(x) \wedge Q(x)\big)\Big) \equiv \forall\, x, \sim f(x) \vee \sim Q(x) \equiv \forall x\ (f(x)\ \longrightarrow \sim Q(x))$
$\equiv$ No $f(x)$ are $Q(x)$

Example 1.6.14.

(i) Express each of the following sentences in the form $\forall x, P(x)$ and then give its negation in both cases $\forall x, P(x)$ and in words.

r: The square of every real number is non-negative.

Solution.

- **$\forall x, P(x)$ form:** r: $\forall x \in \mathbb{R}, x^2 \geq 0$.
- **Negation:** ~r: $\sim (\forall x \in \mathbb{R}, x^2 \geq 0) \equiv \exists x \in \mathbb{R}, \sim (x^2 \geq 0) \equiv \exists x \in \mathbb{R}, x^2 < 0$.
- **Negation in words:** ~r:There exists a real number whose square is negative.

(ii) Let **r: Student who is intelligent will succeed**. Write “ r ” in predicate logic and English logic, and then give its negation in both cases.

Solution.

Let P: Student
Q: intelligent.
S: Succeed

- **Predicate Logic:** r: $\forall x((\mathrm{P}(x) \wedge \mathrm{Q}(x)) \rightarrow \mathrm{S}(x))$
- **Negation:** ~r: $\sim \left[\forall x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \rightarrow \mathrm{S}(x)\right)\right]$
$\equiv \sim \left[\forall x \left(\sim \left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \vee \mathrm{S}(x)\right)\right]$ Implication Low.
$\equiv \exists x \left(\left(\mathrm{P}(x) \wedge \mathrm{Q}(x)\right) \wedge \sim \mathrm{S}(x)\right)$ De Mover’s Law.
- **English logic:** ~r: There exist student who is intelligent and not succeed.

(iii) r: Some integer numbers are even but not odd.

Let $\mathbb{Z} \coloneqq$ **Set of Integers**, $f :=$ **is even**, $P :=$ **is odd**.

- **Predicate Logic:** r : $\exists x \in \mathbb{Z}, \left(f(x) \wedge \sim P(x)\right) \equiv \sim \left[\forall x \left(f(x) \longrightarrow P(x)\right)\right]$.
- **English Logic:** r: Not all even integers are odd.
- **Negation:** ~r: $\sim\sim \left[\forall x \left(f(x) \longrightarrow Q(x)\right)\right] = \left[\forall x \left(f(x) \longrightarrow Q(x)\right)\right]$.
- **Negation in words:** All even integer numbers are odd.

Remark 1.6.15.

Sometimes the English sentences are **unclear** with respect to quantification, or in another wards, quantified statements are often misused in **casual (informal) conversation**.

For example:

(i) “If you can solve any problem we come up with, then you get an A for the course”

The phrase **"you can solve any problem we can come up with"** could reasonably be interpreted as either a universal or existential quantification:

(a) "you can solve every problem we come up with",

(b) "you can solve at least one problem we come up with".

(ii) r**:** All students do not pay full tuition.
Here " r " could reasonably be interpreted as
(a) Not all students pay full tuition (Or: There exist some students do not pay full tuition).
(b) No students are pay full tuition (Or: There are no students are pay full tuition).
Mathematical context: Not all students pay full tuition.

(iii) r: All integer numbers are not even."
(a) Not all integer numbers are even.
(b) No integer numbers are even (Or: There are no even integers).
Mathematical context: Not all integer numbers are even.

Combined Quantifiers 1.6.16. There are six ways in which the quantifiers can be combined when two variables are present:

(1) $\forall\, x\, \forall\, y, f(x,y) \equiv\ \forall\, y\, \forall\, x, f(x,y)$=For every x, for every y, $f(x,y)$.

(2) $\forall\, x\, \exists\, y, f(x,y)$ ≡ For every x, there exists a y such that $f(x,y)$.

(3) $\forall\, y\, \exists\, x, f(x,y)$ ≡ For every y, there exists an x such that $f(x,y)$.

(4) $\exists\, x\, \forall\, y, f(x,y)$ ≡There exists an x such that for every y, $f(x,y)$.

(5) $\exists\, y\, \forall\, x\ f(x,y)$ ≡ There exists a y such that for every x, $f(x,y)$.

(6) $\exists\, x\, \exists\, y, f(x,y) \equiv\ \exists\, y\, \exists\, x, f(x,y)$= There exists an x such that there exists a y, $f(x,y)$.

Example 1.6.17.

(i) r: $\exists\, x \in \mathbb{R}\, \exists\, y \in \mathbb{R} : P(x,y) = (x^2 + y^2 = 2xy)$. The proposition " r " is true since $x = y = 1$ is one of many solutions.