Example 1.6.17.

(i) r: $\exists\, x \in \mathbb{R} \,\exists\, y \in \mathbb{R} : P(x,y) = (x^2 + y^2 = 2xy)$. The proposition " r " is true since $x = y = 1$ is one of many solutions.

(ii) s: $\forall\, x \in \mathbb{R}\ \exists\, y \in \mathbb{R} : P(x,y) = (y^3 = x)$. The proposition " s " is true since $y = \sqrt[3]{x}$ is solution for $P(x,y)$.

(iii) s: $\exists\, x \in \mathbb{R}\ \ \forall y \in \mathbb{R} : P(x,y) = (y^3 = x)$. Here "s" mean there is an "x" real such that for every "y" real, $P(x,y)$ is true. The proposition " s " is not true since no real numbers have this property.

(iv) r: For all x, there exists y such that $xy = 1$.

Solution.

- **$\forall x, P(x)$ form:** r: $\forall\, x$, $\exists\, y$ such that $xy = 1$.
- **Negation:** ~r: ~ ($\forall\, x$, $\exists\, y$ such that $xy = 1$)
  $\equiv\ \exists x$, ~ ($\exists\, y$ such that $(xy = 1)$)
  $\equiv\ \exists x$, $\forall\, y$ such that $xy \neq 1$.
- **Negation in words:** ~r: There exists x such that for all y, $xy \neq 1$.

(v) The following are equivalents.

(a) $\sim[\forall\, x\, \forall\, y, f(x,y)] \equiv \exists x\ \ \exists y, \sim f(x,y)$.

(b) $\sim[\exists\, x\ \ \exists y, f(x,y)] \equiv \forall x\ \ \forall y, \sim f(x,y)$.

(c) $\sim[\forall\, x\, \exists\, y, f(x,y)] \equiv \exists x\ \ \forall y, \sim f(x,y)$.

(d) $\sim[\exists\, x\, \forall\, y, f(x,y)] \equiv \forall x\ \ \exists y, \sim f(x,y)$.

Solution. Exercise.

1.7. Logical Reasoning

Definition 1.7.1. (Arguments)
An **argument** is a series of statements starting from hypothesis (premises/ assumptions) and ending with the conclusion.

From the definition, an argument might be valid or invalid.

Definition 1.7.2. (Valid Arguments)(Proofs)
An argument is said to be **valid** if the hypothesis implies the conclusion; that is, if s is a statement implies from the statements $s_1, s_2, \dots, s_n$, then write as

$$s_1, s_2, \dots, s_n \mapsto s.$$

Note 1.7.3. In mathematics, the word ***proof*** is used to mean simply a valid argument. Many proofs involve more than two premises and a conclusion.

Example 1.7.4.
(i) Let s_1: Some mathematicians are engineering
s_2 : Ali is mathematician
s : Ali is engineering
Show that the argument is valid.
Solution.
The argument $s_1, s_2 \mapsto s$ is not valid, since not all mathematicians are engineering.

(ii) Let s_1: There is no lazy student
s_2 : Ali is artist
s_3 : All artist are lazy
Find a conclusion s for the above premises making the argument $s_1, s_2, s_3 \mapsto s$ is valid.
Solution.
Ali is**----------------------.**

Remark 1.7.5.
(i) An argument $\boxed{s_1, s_2, \dots, s_n \mapsto s}$
is valid if and only if $\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \to s}$ is tautology; that is,

$$\boxed{(s_1 \wedge s_2 \wedge \dots \wedge s_n) \Longrightarrow s.}$$

(ii) An argument does not depend on the truth of the premises or the conclusion but it just interested only in the question

"**Is the conclusion implied by the conjunction of the premises?**"

Example 1.7.6. Show that the following argument is valid using truth table.

A_1: $p \wedge q$

A_2: $p \to \sim (q \wedge r)$

A_3: $s \to r$

C: $\therefore \sim s$

				A_1			A_2	A_3
p	**q**	**r**	**s**	**p ∧ q**	**(q ∧ r)**	**~(q ∧ r)=I**	**p→ I**	**s→ r**
T	T	T	T	T	T	F	F	T
T	T	T	F	T	T	F	F	T
T	T	F	T	T	F	T	T	F
T	T	F	F	T	F	T	T	T
T	F	T	T	F	F	T	T	T
T	F	T	F	F	F	T	T	T
T	F	F	T	F	F	T	T	F
T	F	F	F	F	F	T	T	T
F	T	T	T	F	T	F	T	T
F	T	T	F	F	T	F	T	T
F	T	F	T	F	F	T	T	F
F	T	F	F	F	F	T	T	T
F	F	T	T	F	F	T	T	T
F	F	T	F	F	F	T	T	T
F	F	F	T	F	F	T	T	F
F	F	F	F	F	F	T	T	T

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	A_1	A_2	A_3	$A_1 \wedge A_2 \wedge A_3$	C	$(A_1 \wedge A_2 \wedge A_3) \to$ **C**	
s	**p ∧ q**	**p→ I**	**s→ r**		~s		
T	T	F	T	F	F	T	
F	T	F	T	F	T	T	
T	T	T	F	F	F	T	
F	**T**	**T**	**T**	**T**	**T**	**T**	⟵
T	F	T	T	F	F	T	
F	F	T	T	F	T	T	
T	F	T	F	F	F	T	
F	F	T	T	F	T	T	
T	F	T	T	F	F	T	
F	F	T	T	F	T	T	
T	F	T	F	F	F	T	
F	F	T	T	F	T	T	
T	F	T	T	F	F	T	
F	F	T	T	F	T	T	
T	F	T	F	F	F	T	
F	F	T	T	F	T	T	

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1.8. Mathematical Proof

In this section some common procedures of proofs in mathematics are given with examples.

1.8.1 To Prove Statement of Type (p → q).
(1) Rule of conditional proof.
Let p is true statement and $s_1, s_2, \dots, s_n$ all previous axioms and theorems. To prove p → q it is enough to prove

$$s_1, s_2, \dots, s_n, p \mapsto q$$

is valid argument.

Example 1.8.2. Prove that, a is an even number → a^2 is an even number.
Proof.
Suppose a is an even number.
(1) $a = 2k$, k is an integer (definition of even number).
(2) $a^2 = 4k^2$, square both sides of (1)
(3) $a^2 = 2(2k^2)$, Common factor
(4) a^2 is even number, since $2k^2$ is an integer and definition of even number.

Note that in the above proof, we proved the tautology

$$(s_1 \wedge s_2 \wedge p) \rightarrow q$$

where
p: a is an even number
s_1: $a = 2k$,
s_2: $a^2 = 4k^2$,
q: a^2 is even number.

(2) Contrapositive
To prove p → q we can proof that (∼ q → ∼p) since (p → q) ≡ (∼ q → ∼p).

Example 1.8.3. Prove that, $a^2$is an even number → a is an even number.
Proof.
Let p: $a^2$is an even number,
q: a is an even number.
Then
∼p: $a^2$is an odd number,
∼q: a is an odd number.
Therefore, the contrpositive statement is

a is an odd number $\rightarrow$ a^2 is an odd number.
(1) $a = 2k + 1$ k is an integer (Definition of odd number)
(2) $a^2 = 4k^2 + 4k + 1$ Square both sides of (1)
(3) $a^2 = 2(2k^2 + 2k) + 1$
(4) a^2 is odd number since $2k^2 + 2k$ is an integer and definition of odd number.

Prove Statement of Type (p ⟷ q). 1.8.4.
(i) Since (p ⟶ q) ∧ (q ⟶ p) ≡ (p ⟷ q), so we can proved first p ⟶ q and then proved q ⟶ p.
(ii) Moved from p into q through series of logical equivalent statements s_i as follows:

$$p \leftrightarrow s_1$$
$$s_1 \leftrightarrow s_2$$
$$\vdots$$
$$s_{n-1} \leftrightarrow s_n$$
$$s_n \leftrightarrow q$$

This is exactly the tautology

$$((p \leftrightarrow s_1) \wedge (s_1 \leftrightarrow s_2) \wedge \ldots \wedge (s_n \leftrightarrow q)) \rightarrow (p \leftrightarrow q).$$

Prove Statement of Type $\forall x\, P(x)$ or $\exists x\, P(x)$. 1.8.5.

(i) To prove a sentence of type $\forall\, x\, P(x)$, we suppose x is an arbitrary element and then prove that $P(x)$ is true.
(ii) To prove a sentence of type $\exists\, x\, P(x)$, we have to prove there exist at least one element x such that $P(x)$ is true.

Prove Statement of Type (p ∨ r) ⟶ q. 1.8.6.

Depending on the tautology

$$[(p \to q) \wedge (r \to q)] \to [(p \vee r) \to q]$$

We must prove that p⟶q and r⟶q.

Example 1.8.7. Prove that

$$(a = 0 \vee b = 0) \rightarrow (ab = 0)$$

where a, b are real numbers.
Proof.
Firstly, we prove that $(a = 0) \rightarrow (ab = 0)$.
Suppose that $a = 0$, then $ab = 0.b = 0$.
Secondly, we prove that $(b = 0) \rightarrow (ab = 0)$.
Suppose that $b = 0$, then $ab = a.0 = 0$.

Therefore, the statement $(a = 0 \vee b = 0) \rightarrow (ab = 0)$ is tautology.

Proof by Contradiction 1.8.8.

The contradiction is always false statement whatever the truth values of its components. Proof by contradiction is type of indirect proof.
The way of proof logical proposition **p** by contradiction start by supposing that ∼p and then try to find sentence of type

$$R \wedge\sim R$$

where R is any sentence contain **p** or any pervious theorem or any axioms or any logical propositions.
By this way we can also prove sentences of type $\forall x\, P(x)$ or $\exists x\, P(x)$ or (p → q) or (p ⟹q).

Example 1.8.9. Prove that $(x \neq 0) \Longrightarrow (x^{-1} \neq 0)$, x is real number.
Proof.
Let p: $x \neq 0$,
q: $x^{-1} \neq 0$.
We must prove p ⟹q.
Suppose ∼(p ⟹q) is true.
(1) ∼ (p → q) is tautology Def. of logical implication.
(2) ∼ (p → q)≡ ∼ (∼ p∨q) Implication Law
(3) p ∧ ∼q is tautology, De Morgan's Law
(4) $x \neq 0 \wedge x^{-1} = 0$.
(5) $x \cdot x^{-1} = 1 \neq 0$.
(6) $x \cdot x^{-1} = x \cdot 0 = 0$.
(7) $1 = 0$, Inf. (5), (6).
(8) This is contradiction, since $(1 \neq 0) \wedge (1 = 0)$ Contradiction Law
Thus, the statement ∼(p ⟹q) is not true. Therefore, p ⟹q.

Application Example:
Cryptography (التشفير)

A	**1 0 1 0 0**	**Q**	**1 0 0 1 1**
B	**0 0 0 1 0**	**R**	**1 0 0 1 0**
C	**1 0 1 0 1**	**S**	**1 0 0 0 0**
D	**0 0 1 1 0**	**T**	**0 1 1 1 0**
E	**1 0 1 1 0**	**U**	**0 0 0 1 1**
F	**1 0 1 1 1**	**V**	**0 1 1 0 1**
G	**1 1 0 0 0**	**W**	**0 1 1 1 1**
H	**1 1 0 1 0**	**X**	**0 0 1 0 0**
I	**0 0 0 0 1**	**Y**	**0 1 1 0 0**
J	**1 1 0 0 1**	**Z**	**1 0 0 0 1**
K	**0 0 1 1 1**	**Space**	**1 1 1 1 1**
L	**0 1 0 1 1**	**0**	**1 1 0 1 1**
M	**0 1 0 1 0**	**1**	**1 1 1 0 0**
N	**0 1 0 0 1**	**2**	**1 1 1 0 1**
O	**0 1 0 0 0**	**3**	**1 1 1 1 0**
P	**0 0 1 0 1**	**4**	**0 0 0 0 0**

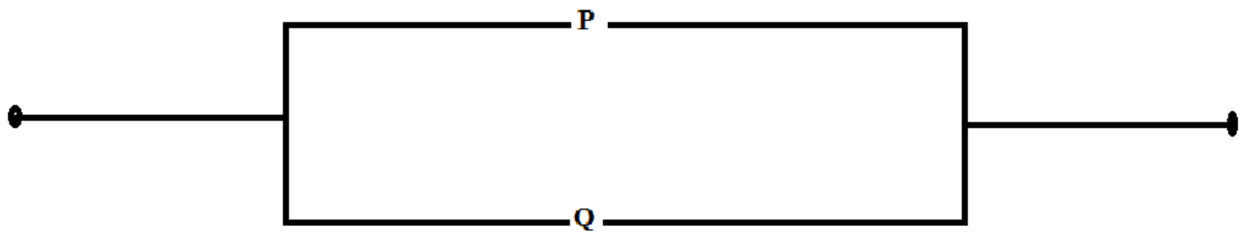
Key: 0 0 1 0 1 0 1 1 0 0 1 1 0 1 0 1 0 1 1 1 1 0 0 0

Plaintext: GO HOME

Plaintext	**G**	**O**		**H**	**O**	**M**	**E**
Code	**11000**	**01000**	**11111**	**11010**	**01000**	**01010**	**10110**
Key	**00101**	**01100**	**11010**	**10111**	**10000**	**01010**	**11001**
XOR							
Encryption	**11101**	**00100**	**00101**	**01101**	**11000**	**00000**	**01111**
Ciphertext	**2**	**X**	**P**	**V**	**G**	**4**	**W**

Ciphertext	**2**	**X**	**P**	**V**	**G**	**4**	**W**
Code	**11101**	**00100**	**00101**	**01101**	**11000**	**00000**	**01111**
Key	**00101**	**01100**	**11010**	**10111**	**10000**	**01010**	**11001**
XOR							
Decryption	**11000**	**01000**	**11111**	**11010**	**01000**	**01010**	**10110**
Plaintext	**G**	**O**		**H**	**O**	**M**	**E**

(ii)



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Exercise

Q1: Show that

(1) $(p \to q) \land \sim q \implies \sim p$.
(2) $p \land (p \to q) \to \sim q$ is a contingency using a truth table.
(3) $p \to (p \lor q)$ is a tautology using a truth table.
(4) $(p \land q) \to p$ is a tautology using a truth table and logical equivalences.
(5) $(p \land q) \to (p \lor q)$ is a tautology using a truth table and logical equivalences.
(6) $[p \to (q \to r)] \equiv [(p \land q) \to r]$ using a truth table and logical proposition.
(7) $[p \to (q \to r)] \equiv [q \to (p \to r)]$ using a truth table and logical proposition.
(8) $[(p \land q) \to p] \equiv [q \to (p \lor \sim p)]$ using a truth table and logical proposition.
(9) $[(p \to q) \land (p \to r)] \equiv [p \to q \land r)]$ using a truth table and logical proposition.

(10) $[(p \to q) \land (r \to q)] \equiv [(p \lor r) \to q]$ using a truth table and logical proposition.

(11) $(p \to q) \equiv (\sim q \to \sim p)$ using a truth table and logical proposition.

(12) $p \land (\sim p \lor q) \equiv p \land q$ using a truth table and logical proposition.
(13) $p \lor (p \land q) \equiv p$ using a truth table and logical proposition.

(14) Is $\veebar$ commutative or associative?

(15) Is $\veebar$ distributive over $\land$, $\lor$, or $\to$?

(16) Is this true $p \veebar q \equiv p \leftrightarrow \sim q$?

(17) $[(p \to q) \land (q \to r)] \implies (p \to r)$ using a truth table.
(18) $[(p \lor q) \land \sim p]) \implies q$ using a truth table.
(19) $[(p \to q) \land (r \to s)] \implies [(p \lor r) \to (q \lor s)]$ using a truth table.

(20) $[(p \to q) \land (p \lor r)] \implies (q \lor r)$ using a truth table.

Q2: Given the hypotheses:

(i) "It is not sunny this afternoon and it is colder than yesterday"
(ii) "We will go swimming only if it is sunny"
(iii) "If we do not go swimming, then we will take a canoe trip"
(iv) "If we take a canoe trip, then we will be home by sunset"
Does this imply that **"we will be home by sunset"?**

Q3: Represent as propositional expressions, and use De Morgan's Laws to write the negation of the expression, and translate the negation in English.

"Tom is a math major but not computer science major."

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Q4: Let

p = "John is healthy"
q = "John is wealthy"
r = "John is wise"

Represent symbolically:

(i) John is healthy and wealthy but not wise.

(ii) John is not wealthy but he is healthy and wise.

(iii) John is neither healthy nor wealthy nor wise.

Q5: Translate the sentences into propositional expressions:

"Neither the fox nor the lynx can catch the hare if the hare is alert and quick."

Q6: Represent as propositional expressions.

"You can either (stay at the hotel and watch TV) or (you can go to the museum and spend some time there)".

Q7: Given a sentence **"If we are on vacation, we go fishing."** Then

(i) translate the sentence into a logical expression,

(ii) write the negation of the logical expression and translate the negation into English,

(iii) write the converse of the logical expression and translate the converse into English,

(iv) write the inverse of the logical expression and translate the inverse into English,

(v) write the contrapositive of the logical expression and translate the contrapositive into English.

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Q8: Write the contrapositive, converse and inverse of the expressions:

$p \rightarrow q$,

$\sim p \rightarrow q$,

$q \rightarrow \sim p$.

Q9: Determine whether the following arguments are valid or invalid:

(i) Premises:

(a) If I read the newspaper in the kitchen, my glasses would be on the kitchen table.

(b) I did not read the newspaper in the kitchen.

Conclusion: My glasses are not on the kitchen table.

(ii) Premises:

(a) If I don't study hard, I will not pass this course

(b) If I don't pass this course I cannot graduate this year.

Conclusion: If I don't study hard, I won't graduate this year.

(iii) Premises:

(a) You will get an extra credit if you write a paper or if you solve the test problems.

(b) You don’t write a paper, however you get an extra credit.

Conclusion: You have solved the test problems.

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Q10: Find an expression equivalent to $p \rightarrow q$ that uses only $\wedge$ and $\sim$.

Q11: Negate the following sentences.

(i) The number x is positive, but the number y is not positive.
(ii) If x is prime, then $\sqrt{x}$ is not a rational number.
(iii) For every prime number p, there is another prime number q with $q > p$.
(iv) There exists a real number a for which $a + x = x$ for every real number x.

(v) Every even integer greater than 2 is the sum of two primes.
(vi) The integer x is even, but the integer y is odd.
(vii) At least one of the integers x and y is even.
(viii) The numbers x and y are both odd.
(ix) For every real number x there is a real number y for which $y^3 = x$.
(x) I don't eat anything that has a face.

Q12: Write the following propositions with quantifiers and then give its negation with translations into words.

(i) Some counting numbers are greater than five

(ii) Every element of set D is less than 7.

(iii) Some elements of set D are less than 13.

(iv) Every counting number greater than 4 is greater than 2.
(v) Some counting numbers are even.
(vi) Every counting number which is divisible by 2 is even.
(vii) Every counting number is even or odd.

(viii) For every x in D_x and for every $y \in$ in D_x, x plus y less than 3.

(ix) At least one politician isn't a logician.

(x) Only nonlogicians are politicians.

Q13: Prove that $\sqrt{2}$ is irrational using contradiction method.

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