محاضرة/رقم (3) (١)

2- Fixed point Iteration method

الطريقة التكرارية للنقطة الصامدة

If the equation $f(x)=0$ whose roots are to be found is expressed as $x=g(x)$ ---- (*)

Then an iterative method, which is very easy to program for a computer may be formulated.

i.e A number p is a fixed point for a given function g if $g(p)=p$. For example the equation $f(x)=e^{-x}-2x+1=0$ may be rewritten as $x=\dfrac{e^{-x}+1}{2}$

The iterative technique is to gues a starting value x_0 substitute it in $g(x)$ to give new approximation $x_1=g(x_0)$. The new approximation is again substituted in $g(x)$ to give a further approximation $x_2=g(x_1)$, and so on until a sufficiently accurate approximation to the root is obtained. This repetitive process, based on $x_{n+1}=g(x_n)$, is called simple iteration; provided that $|x_{n+1}-x_n|$ decreases as n increases.

في هذه الطريقة نكتب المعادلة $f(x)=0$ بالصيغة $x=g(x)$

ان اي نقطة صامدة للدالة g (النقطة x التي تحقق المعادلة (*) مثلا) نقطة صامدة للدالة g تعتبر جذر للمعادلة $f(x)=0$ ولايجاد هذا الجذر نختار اولا قيمة تقريبية اولية x_0 نعوضها نحسب قيمة الدالة g في x_0 للحصول على تقريب اخر للجذر ولكن x_1 اي ان $x_1=g(x_0)$

ثم تكرر العملية لنحصل على تقريب جديد $x_2=g(x_1)$

وهكذا يمكن توليد متتالية من القيم التقريبية للجذر بتطبيق الصيغة التكرارية $x_{n+1}=g(x_n)$

② Ex) Find the root of the equation $3xe^{x}=1$ to an accuracy of $\underline{0.0001}$ (ϵ), using the method of simple iteration with initial point $\underline{x_0=1}$

Solution we first set $x=\frac{1}{3}e^{-x}=g(x)$, $\because x_0=1$

$x_1=\frac{1}{3}e^{-1}=0.12263$

$x_2=0.29486$

$x_3=0.24821$

$x_4=0.26007$

$x_5=0.25700$

$x_6=0.25779$

$x_7=0.25759$

$\boxed{x_8=0.25764}$

$\therefore |x_{n+1}-x_n|=|0.25764-0.25759|$

$=0.00005<0.0001$ (ϵ)

and the root is 0.25764

الفرق أقل من ϵ

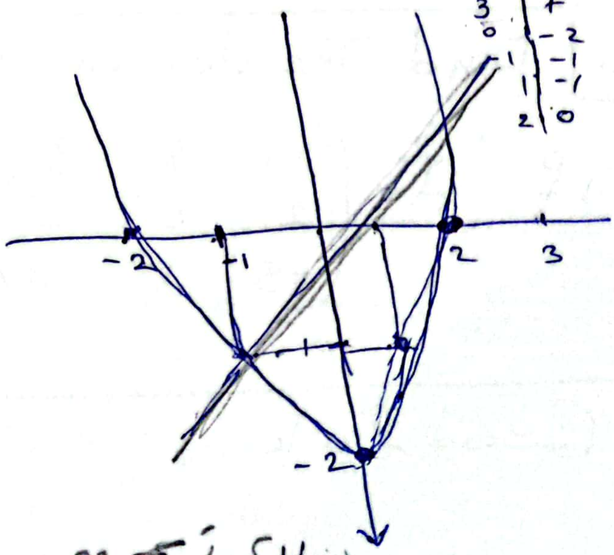
هذا الجذر

Ex) The function $g(x)=x^2-2$, for $-2\le x\le 3$, has fixed points at $x=-1$ and $x=2$

$\because g(-1)=(-1)^2-2=-1$

$g(2)=(2)^2-2=2$

x	$g(x)$
-2	0
3	7
0	-2
-1	-1
1	-1
2	0



$g(x)=x^2-2$

$x=x^2-2$

نرسم هذه مستقيم

هنا نبحث تقاطع

لذلك نرسم مع $y=x$ دائماً

Theorem

(a) If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$

(b) If, in addition, $g'(x)$ exist on (a,b) and a positive constant $\boxed{k<1}$ exists with $|g'(x)| \le k$ for all $x \in (a,b)$ then the fixed point in $[a,b]$ is unique.

Theorem (Fixed_point Theorm)

Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all x in $[a,b]$, suppose, in addition, that g' exists on (a,b) and that a constant $0<k<1$ exist with

$|g'(x)| < k$, $\forall x \in (a,b)$ لازم تكون مقبولة حسب شرط النظرية

Then, for any number P_0 in $[a,b]$, The sequence defined by $P_n = g(P_{n-1})$, $n \ge 1$ converges to the unique fixed point p in $[a,b]$

مما سبق نستطيع القول بأن الشرط الكافي لتقارب الصيغة التكرارية $x_{n+1} = g(x_n)$ هو أن $|g'(x)| < k < 1$

لكل x تقع في جوار الجذر.

الجذر الحقيقي

EX Find a real root of the equation $\cos x = 3x - 1$ correct to three decimal places using Fixed point theorem (to show that Fixed-point Theorem in hold)

لثلاث مراتب عشرية

باستخدام النظرية

Solution Let $f(x) = \cos x - 3x + 1 = 0$

since $f(0) = \cos(0) - 3(0) + 1 = 2$

④

$f(\frac{\pi}{2}) = \cos(\frac{\pi}{2}) - 3(\frac{\pi}{2}) + 1 = -3.712389$

The root lies between 0 and $\frac{\pi}{2}$ $\longrightarrow [0, \frac{\pi}{2}]$

فترة لانه
تكون نته عريب الاصابب

$x = \frac{1}{3}(\cos x + 1) = g(x)$

فترة مغلوقة حسب شرط التقارب

$\Rightarrow g'(x) = -\frac{\sin x}{3}$ & $|g'(x)| = \frac{1}{3}|\sin x| < 1$ in $(0, \frac{\pi}{2})$

$\frac{1}{3}|\sin 0| = 0 < 1$

$|\sin x| < 1$ ليش لأن sin محصورة

$\frac{1}{3}|\sin \frac{\pi}{2}| = \frac{1}{3} < 1$

Hence the fixed-point method can be applied & we start with $x_0 = 0$

$x_1 = g(x_0) = \frac{1}{3}(\cos 0 + 1) = 0.6667$

$x_2 = g(x_1) = \frac{1}{3}(\cos 0.6667 + 1) = 0.5953$

$x_3 = g(x_2) = \frac{1}{3}(\cos 0.5953 + 1) = 0.6093$

$x_4 = g(x_3) = \frac{1}{3}(\cos 0.60931 + 1) = 0.6064$

$x_5 = g(x_4) = \frac{1}{3}(\cos 0.6064 + 1) = 0.6072$

$x_6 = g(x_5) = \frac{1}{3}(\cos 0.6072 + 1) = 0.6071$

we can see that $|x_6 - x_5| = |0.6071 - 0.6072| = 0.0$

hence the root is 0.6071 correct to Three decimal pl

ثلاث خانات بعد الفاصلة متساوية

Example Equation $x^3 + 4x^2 - 10 = 0$, has a unique root in $[1, 2]$, There are many ways to change the equation to the fixed point form $x = g(x)$ using simple algebraic manipulation, and show that the fixed-point

الجبري التلاعب

Theorem hold or not in each ways.

Solution $\because f(1)=1^3+4(1)^2-10=-5$ & $f(2)=2^3+4(2)^2-10=+14$ (5)

سالبة موجبة يوجد جذر

(a) $x = x+x^3+4x^2-10 \rightarrow g_1(x) = x-x^3-4x^2+10$

We have $g_1(1)=\underline{6}$ & $g_1(2)=\underline{-12}$ so g_1 does not ~~map [1,2]~~

map $[1,2] \rightarrow [1,2]$

$|1-3-8| = |-10| = 10 > 1$

Moreover, $g_1'(x) = 1-3x^2-8x$ so $|g_1'(x)|>1$, $\forall x \in (1,2)$

$\Rightarrow$ Theorem fixed-point dose not hold for This choice of g_1 There is no reason to expect convergence.

مافي تقارب

(b) $x^3 = 10-4x^2 \longrightarrow x^2 = \frac{10-4x^2}{x} \rightarrow \boxed{x = \mp\sqrt{\frac{10-4x^2}{x}}}$

نختار الموجب فقط

$\Rightarrow g_2(x) = \left(\frac{10-4x^2}{x}\right)^{\frac{1}{2}}$

$g_3(x) = \frac{1}{2}\sqrt{10-x^3}$, $g_4(x) = \sqrt{\frac{10}{(4+x)}}$

$g_5(x) = x - \frac{x^3+4x^2-10}{3x^2+8x}$.

$\rightarrow$ $4x^2 = 10-x^3$

$x^2 = \frac{10-x^3}{4}$

$x = \mp\frac{\sqrt{10-x^3}}{2} \longrightarrow$ تأخذ الموجب

$g_4 = x^2(x+4) = 10$

$x^2 = \frac{10}{x+4} \Rightarrow x = \mp\sqrt{\frac{10}{x+4}}$