26
DOUBLE REFRACTION

From the standpoint of physical optics, doubly refracting crystals are classified as either *uniaxial* or *biaxial*. We have seen that in uniaxial crystals the refractive indices, and hence the velocities, of the *O* and *E* waves become equal along a unique direction called the optic axis. In biaxial crystals, on the other hand, there are two directions in which the velocity of plane waves is independent of the orientation of the incident vibrations. These two optic axes make a certain angle with each other which is characteristic of the crystal and depends to some extent on the wavelength. Uniaxial crystals may be thought of as a special case of biaxial crystals where the angle between the axes is zero.

26.1 WAVE SURFACES FOR UNIAXIAL CRYSTALS

Uniaxial crystals may be divided into two classes, *negative* and *positive*. In a negative crystal like calcite, the extraordinary index of refraction is less than the ordinary index. In quartz, a positive crystal, the index of the extraordinary ray is greater than that of the ordinary ray. A general treatment of the propagation of light in positive

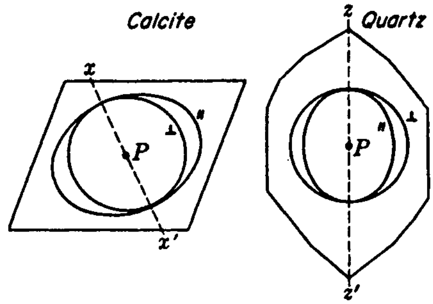

FIGURE 26A
Wave-surface diagrams in calcite and quartz crystals.

and negative crystals is usually given in terms of wave surfaces, which lend themselves so aptly to Huygens' construction.

The wave surface is a wave front (or pair of wave fronts) completely surrounding a point source of monochromatic light. Thus if in one of the crystals of Fig. 26A the source is at P, the circle and ellipse around it represent the traces of the wave fronts, which are the loci of points of *equal phase* in the waves given out by P. If these crystals were isotropic substances such as glass, there would be a single wave surface, which would take the form of a sphere, showing that the velocity of the wave in all directions is the same. In most crystalline substances, however, two wave surfaces are formed, one called the *ordinary wave surface* and the other the *extraordinary wave surface*. In both calcite and quartz the ordinary wave surface is a sphere and the extraordinary wave surface an ellipsoid of revolution. The actual three-dimensional surfaces are obtained by rotating the cross-sectional figures of Fig. 26A about the optic axes which, for reasons to be explained, are labeled xx' and zz'. The circle generates a sphere, and the ellipse an ellipsoid of revolution. The three cross sections of these surfaces are shown in Fig. 26B. The eccentricity of the elliptical sections in these figures is exaggerated, the major and minor axes actually differing by only 11 percent for calcite and 0.6 percent for quartz.

In calcite the ellipsoid touches the enclosed sphere at the two points where the optic axis through P passes through the surfaces. In quartz the sphere and enclosed ellipsoid do not quite touch at the optic axis through P. The fact that they do not touch gives rise to a whole new phenomenon called *optical activity*, a subject which will be treated in detail in Chap. 28. The approach of the two surfaces along the optic axis is so close, however, that for the present they will be assumed to touch as they in fact do in other positive crystals like titanium dioxide, zinc oxide, ice, etc. It should be pointed out that owing to the dispersion of all media the wave surfaces shown apply to one wavelength only. Correspondingly smaller or larger surfaces would be drawn for other wavelengths. Furthermore, it is important to remember that the radii drawn from P are proportional to phase velocities and hence do not measure the rate of propagation of energy. The group velocities, which in dispersive media are usually less than phase velocities (Sec. 23.7), would be represented by proportionately smaller surfaces. They would be the same as the wave surfaces described here only in the case of ideal monochromatic light.

The directions of vibration in the two wave surfaces are indicated in Fig 26A

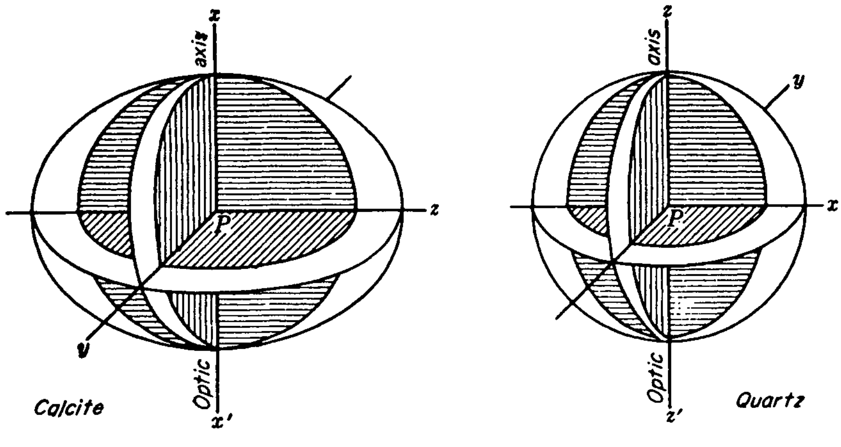

FIGURE 26B
Cross sections of the wave surfaces for calcite and quartz crystals.

by $\perp$ for vibrations perpendicular to the page and by $\|$ for vibrations in the plane of the page. These will be more exactly specified after we have considered how the wave surfaces may be applied.

26.2 PROPAGATION OF PLANE WAVES IN UNIAXIAL CRYSTALS

The origin of the double refraction of light at a crystal surface is readily explained in terms of the wave surfaces just described. This is accomplished by the use of Huygens' principle of secondary wavelets. Consider, for example, a beam of parallel light incident normally on the surface of a crystal like calcite, whose optic axis makes some arbitrary angle with the crystal surface (see Fig. 26C). The optic axis has the direction shown by the broken lines. According to Huygens' principle, we may now choose points anywhere along the wave front as new point sources of light. Here A, B, and C are chosen just as the wave strikes the crystal boundary. After a short time interval the Huygens wavelets entering the crystal from these points will have the form shown in the figure.

If one now proceeds to find the common tangents to these secondary wavelets, the result is the two plane waves labeled OO' and EE' in the figure. Since the first is tangent to spherical wavelets, it behaves like a wave in an isotropic substance, traveling perpendicular to the surface with a velocity proportional to AA', BB', and CC'. We have seen in the last chapter that the vibrations for this O wave are normal to the principal section. The tangent to the ellipsoidal wavelets represents the wave front for the E vibrations, which take place in the principal section. The E rays, connecting the origins of the wavelets with the points of tangency, diverge from the O rays and are no longer perpendicular to the wave front. They represent the direc-

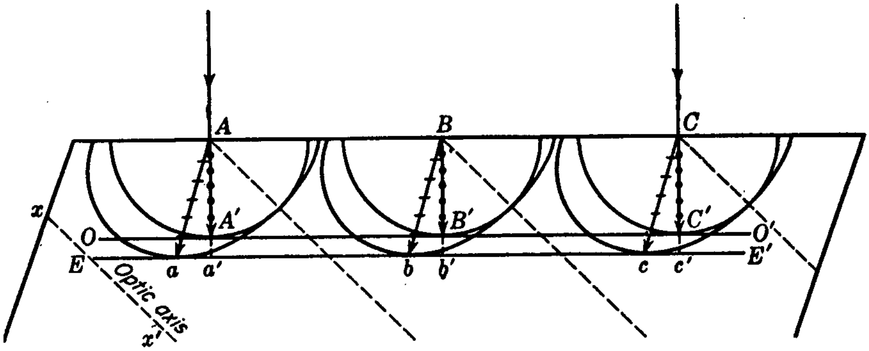

FIGURE 26C
Huygens' construction for a plane wave incident normally on a calcite crystal.

tion in which a narrow beam of light would be refracted, which is the direction in which the energy of the E vibrations is transmitted. Its velocity, proportional to Aa, Bb, or Cc, is called the *ray velocity*. This is greater than the normal velocity, measured by Aa', Bb', or Cc', the velocity with which the wave advances through the crystal in a direction normal to its own plane.

If the normal velocity Aa' is plotted in polar coordinates as a function of the angle between the optic axis and the E-wave normal, we get the dashed ovals of Fig. 26D. These ovals are of course three-dimensional surfaces symmetrical about the optic axis. Now it is seen that the wave surface, i.e., the ellipsoid of revolution, is really a ray-velocity surface. The normal-velocity surface and the ray-velocity surface for the ordinary vibrations are both represented by the same circle or sphere. Hereafter the ellipsoid of revolution will be referred to as the *wave surface* of the E wave and the oval of revolution as the *normal-velocity surface* of the E wave.

In constructing Fig. 26C the optic axis was assumed to be in the plane of the

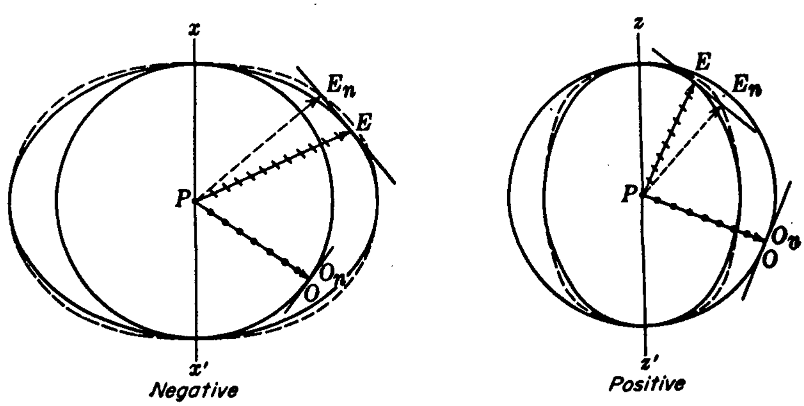

FIGURE 26D
Wave surfaces and normal-velocity surfaces for uniaxial crystals.

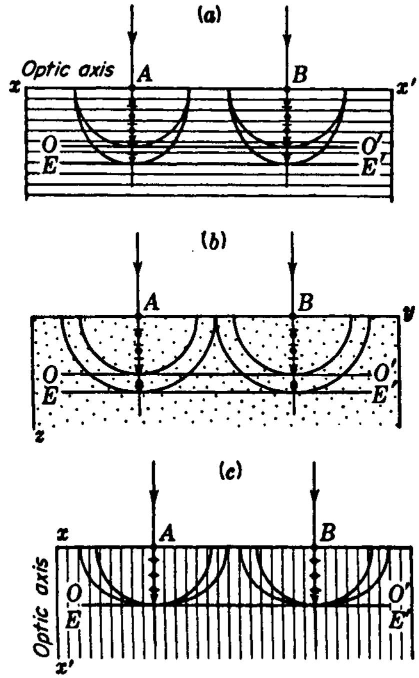

FIGURE 26E
Propagation of normally incident plane waves through calcite crystals cut parallel and perpendicular to the optic axis.

page. In case the optic axis is not in the plane of the page, a plane drawn tangent to the ellipsoidal wavelets will make contact at points in front of or in back of the plane of the page. If the optic axis is either parallel or perpendicular to the surface of the crystal, however, the situation is especially simple. Figure 26E illustrates Huygens' construction in these important cases, where the crystal face is cut (1) parallel to the optic axis as in (*a*) and (*b*), and (2) perpendicular to the optic axis as in (*c*). In both cases the ray velocities are equal to the normal velocities, and there is no double refraction. In case 1, however, the E wave travels faster than the O wave. When a difference in these velocities exists, we obtain the phenomenon of interference in polarized light, discussed in the following chapter.

It will assist in understanding the rather complex behavior of the velocity of light vibrating in different directions, and described by the wave surface, to note the following facts. The O wave, which vibrates everywhere perpendicular to the optic axis, has the same velocity in every direction. The vibrations of the E wave make a different angle with the axis for each different ray that is drawn from P (Fig. 26D). In particular, for the ray drawn along the optic axis, the vibrations of which are perpendicular to the axis, the velocity becomes equal to that of the O ray, which is also vibrating perpendicular to the axis. These facts suggest that the velocity of light is for some reason dependent on the angle of inclination between the vibrations and the optic axis. In terms of the elastic-solid theory, this could be explained by assuming two different coefficients of elasticity for vibrations parallel and perpendicular to the optic axis. In calcite, for example, the restoring force is taken to be greater for the E ray

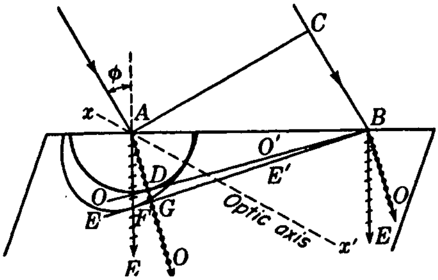

FIGURE 26F
Huygens' construction when the optic axis of a calcite crystal lies in the plane of incidence.

traveling perpendicular to the optic axis (vibrations parallel to the axis) than for the O ray in the same direction (vibrations perpendicular to the axis). Hence the E wave travels faster in this direction.

26.3 PLANE WAVES AT OBLIQUE INCIDENCE

Continuing the study of the double refraction of light in uniaxial crystals, consider the case of a beam of parallel light incident at an angle on the surface of a crystal whose optic axis lies in the plane of incidence and at the same time makes some arbitrary angle with the crystal surface (see Fig. 26F). At the point A where the light first strikes the boundary, the O-wave surface is drawn with such a radius that the ratio CB/AD is equal to the refractive index of the O ray. The ellipsoidal wave surface is then drawn tangent to the circle at the intersection with the optic axis xx'. The points D and F and the new wave fronts DB and FB are located by drawing tangents from the common point B to the circle and ellipse. While the light is traveling from C to B in air, the O vibrations travel from A to D in the crystal and the E vibrations travel from A to F. In the more general case where the optic axis is not in the plane of incidence, the refracted ray will not lie in the same plane. Such cases require three-dimensional figures and cannot be easily shown.

The principles of Huygens' construction are applied to three special cases in Fig. 26G. In (*a*) and (*c*), the optic axis, the plane of incidence, and the E and O principal planes coincide with the plane of the page. In (*b*) the axis is perpendicular to the plane of incidence, and the cross sections of the wave surfaces from A yield two circles. This is a case where the two principal planes defining the directions of vibrations of the O and E rays (Sec. 24.9) are separate from each other and from the principal section.

From geometry it may be shown for the special case of Fig. 26G(*a*), where the optic axis is in the surface as well as in the plane of incidence, that the directions of the refracted rays are given by

$$\frac{n_E}{n_O} = \frac{\tan \phi'_E}{\tan \phi'_O}$$

Here ϕ'_E and ϕ'_O are the angles of refraction, and n_E and n_O are the principal refractive indices.

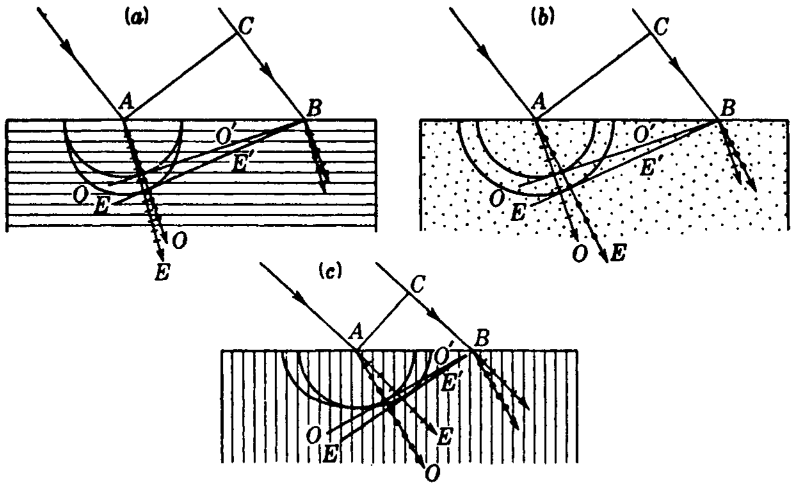

FIGURE 26G
Double refraction for crystals cut with the optic axis parallel and perpendicular to the surface.

26.4 DIRECTION OF THE VIBRATIONS

In crystals the physical nature of the "vibrations" must be specified more closely than merely as the oscillations of the electric (or magnetic) vector used thus far. For reasons to be discussed below, the direction of the electric displacement **D** (Sec. 23.9) is not in general the same as that of the electric field **E**. The application of Maxwell's equations to anisotropic substances along the lines to be outlined in Sec. 26.9 shows that the vibrations lying in the wave front are those of **D**. The vibrations of **E**, however (i.e., of the electric vector—not to be confused with the symbol *E* for the extraordinary wave) are perpendicular to the ray, and hence inclined to the wave front. Hence the extraordinary wave is a transverse wave for **D** but not for **E**. In our Figs. 26C and 26D, and in what follows, we indicate as the direction of the vibrations that of the electric displacement **D**.

For uniaxial crystals, the directions of vibration of the *O* and *E* rays may be specified in terms of the principal planes for these rays defined in Sec. 24.9. The *O* vibrations are perpendicular to the principal plane of the *O* ray, which contains this ray and the optic axis. They are also tangent to the *O*-wave surface. The *E* vibrations lie in the principal plane of the *E* ray and are tangent to the *E*-wave surface. These definitions may seem unnecessarily complicated in cases like Fig. 26C, where the principal section and the two principal planes all coincide with the plane of the figure, but they are essential in the more general case, where all three of these planes are different. Another way of determining the directions of the vibrations, which holds quite generally for all cases including biaxial crystals, is as follows. The electric displacements associated with one ray (the *E* ray in uniaxial crystals) are in the direction of the projection of the ray on its wave front. Those associated with the other ray may then be found from the fact that for a given direction of the wave normal the two possible directions of **D** are mutually perpendicular. Inspection of our figures will show agreement with these rules in the simple cases we have considered.

26.5 INDICES OF REFRACTION FOR UNIAXIAL CRYSTALS

The index of refraction is usually defined as the ratio of the velocity of light in vacuo to the velocity in the medium in question. In uniaxial crystals there are two principal indices of refraction, one expressing the velocity of the E wave when traveling normal to the optic axis, and the other the velocity of the O wave. These are related to the two elastic coefficients mentioned in Sec. 26.2. In negative crystals, such as calcite, *the principal index for the extraordinary wave is defined as the velocity of light in vacuo divided by the maximum velocity in the crystal.*

$$n_E = \frac{\text{velocity in vacuo}}{\text{maximum velocity of } E \text{ wave}} \tag{26a}$$

The maximum normal velocity, it should be noted, is equal to the maximum ray velocity. The ordinary index is defined as

$$n_O = \frac{\text{velocity in vacuo}}{\text{velocity of } O \text{ wave}} \tag{26b}$$

In positive uniaxial crystals the principal index for the extraordinary wave is defined as

$$n_E = \frac{\text{velocity in vacuo}}{\text{minimum velocity of } E \text{ wave}} \tag{26c}$$

The principal indices for calcite and quartz are given in Table 26A for several wavelengths throughout the visible, ultraviolet, and near-infrared spectrum.

Since the E-wave surface touches the O-wave surface at the optic axis, the ordinary index n_O also gives the velocity of the E wave along the axis. Each pair of values of n_O and n_E for a given wavelength therefore determines the ratio between the major and minor axes of the extraordinary wave surfaces for that wavelength of light.

The principal indices for uniaxial crystals are readily determined experimentally by refracting light through a prism of known angle. If either of the prisms in Fig. 26H is placed on a spectrometer table, two spectra will be observed. For any given wavelength there will be two spectrum lines and hence two angles of minimum deviation. The O and E indices are then calculated in the usual way (Sec. 2.5) by the formula

$$n = \frac{\sin \frac{1}{2}(\alpha + \delta_m)}{\sin \frac{1}{2}\alpha} \tag{26d}$$

where δ_m is the angle of minimum deviation and α is the angle of the prism.

At minimum deviation in prism (*a*) the E ray is traveling essentially perpendicular to the optic axis, the necessary conditions for measuring the principal index n_E. In prism (*b*) it should be noted that the cross section of the wave surface yields two

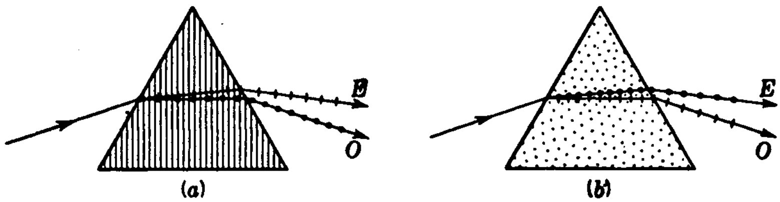

FIGURE 26H
Double refraction in prisms cut from a negative uniaxial crystal.

circles. This means that the velocity of the E ray, as well as that of the O ray, is independent of direction in the plane of the figure and Snell's law of refraction holds for it also.

Two useful relationships for calculating points on an ellipse, plotted in rectangular coordinates, are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \begin{aligned} x &= a\cos\phi \\ y &= b\sin\phi \end{aligned} \tag{26e}$$

One most interesting uniaxial crystal, called *rutile*, is synthesized TiO_2 (titanium oxide), a clear water-white crystal used in making gemstones that show approximately

Table 26A PRINCIPAL INDICES OF CALCITE AND QUARTZ AT 18°C

Source element	Wavelength, Å	Calcite n_O	Calcite n_E	Quartz n_O	Quartz n_E	Fused quartz
Au	2000.60	1.90302	1.57663	1.64927	1.66227	
Cd	2265.03	1.81300	1.54914	1.61818	1.62992	1.52308
Cd	2573.04	1.76048	1.53013	1.59622	1.60714	1.50379
Cd	2748.67	1.74147	1.52267	1.58752	1.59813	1.49617
Sn	3034.12	1.71956	1.51366	1.57695	1.58720	1.48594
Cd	3403.65	1.70080	1.50561	1.56747	1.57738	1.47867
Hg	4046.56	1.68134	1.49694	1.55716	1.56671	1.46968
H_γ	4340.47	1.67552	1.49552	1.55396	1.56340	1.46690
H_β	4861.33	1.66785	1.49076	1.54968	1.55898	1.46318
Hg	5460.72	1.66168	1.48792	1.54617	1.55535	1.46013
Hg	5790.66	1.65906	1.48674	1.54467	1.55379	
Na	5892.90	1.65836	1.48641	1.54425	1.55336	1.45845
H_α	6562.78	1.65438	1.48461	1.54190	1.55093	1.45640
He	7065.20	1.65207	1.48359	1.54049	1.54947	1.45517
K	7664.94	....	....	1.53907	1.54800	
Rb	7947.63	....	....	1.53848	1.54739	1.45340
	8007.00	1.64867	1.48212			
O	8446.70	....	....	1.53752	1.54640	
	9047.0	1.64579	1.48095			
Hg	10140.6	....	....	1.53483	1.54360	
	10417.0	1.64276	1.47982			

6 times the fire of a diamond. The refractive indices given in Table 26B, are calculated from the modified two-term Cauchy equations;

$$\begin{aligned} O \text{ ray:} \quad & n_O^2 = 5.913 + \frac{2.441 \times 10^7}{\lambda^2 - 0.803 \times 10^7} \\ E \text{ ray:} \quad & n_E^2 = 7.197 + \frac{3.322 \times 10^7}{\lambda^2 - 0.843 \times 10^7} \end{aligned} \tag{26f}$$

26.6 WAVE SURFACES IN BIAXIAL CRYSTALS

The majority of crystals occurring in nature are biaxial, possessing two optic axes, or directions of single normal velocity. Double refraction in such crystals, just as in calcite and quartz, is most easily described in terms of wave-surface diagrams and Huygens' principle. Three cross-sectional views of the wave surfaces for a biaxial crystal are given in Fig. 26I. As before, the directions of vibration are shown by dots and lines. Each section cuts the two surfaces in one circle and one ellipse, and these are different in the three sections. The figures are drawn for the case where the semi-axes of the intersections of the wave surface with the coordinate planes are, as shown in the figure, $a = 3$, $b = 2$, and $c = 1$. (Such large differences in a, b, and c are never found in nature.)

Of the three cross sections, the center one (that in the xz plane) is the most interesting, for it contains the four singular points where the outer wave surface (light line) touches the inner surface (heavy line). As shown again in Fig. 26J(a), the rays OR_1 and OR_2 represent directions in which there is but one ray velocity. These are not the optic axes. The optic axes are located by drawing the tangent planes, A_1M_1 and A_2M_2. It is difficult to show in two dimensions that these tangent planes touch the three-dimensional outer surface in circles whose diameters are A_1M_1 and A_2M_2, but such is the case. Since the cross section of one surface is a circle, the lines OA_1 and OA_2 are perpendicular to the tangent planes. They therefore give the same normal velocity for both the ellipse and the circle, so that OA_1 and OA_2 are the optic axes for the point O.

From Fig. 26I it is seen that one can determine the shape of the wave surfaces

Table 26B REFRACTIVE INDICES FOR TiO_2 (RUTILE), FOR SEVERAL OF THE PRINCIPAL FRAUNHOFER LINES

Designation	λ, Å	n_O	n_E
C (H_α)	6561	2.5710	2.8560
D (Na)	5890	2.6131	2.9089
E (Fe)	5270	2.6738	2.9857
F (H_β)	4861	2.7346	3.0631
G′(H_γ)	4340	2.8587	3.2232
H (Ca^+)	3968	3.0128	3.4261

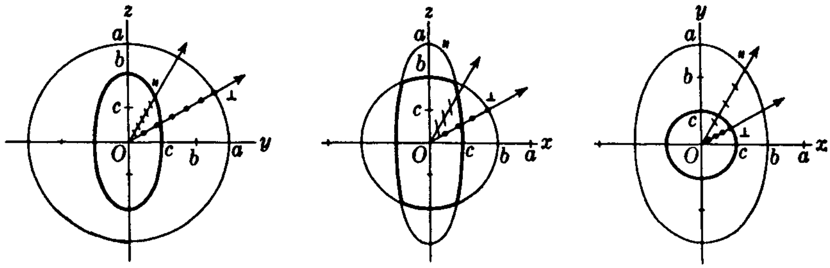

FIGURE 26I
Cross sections of wave surfaces for a biaxial crystal.

by specifying three principal indices of refraction. These are determined by the fact that there are three particular velocities, corresponding to vibrations parallel to x, y, and z, respectively. The elastic-solid theory specified three different coefficients of elasticity for these three types of vibration, which gave rise to these three velocities. If the wave surfaces represent the wave fronts after they have traveled a time of 1 s from the point O, the indices are given by

$$n_a = \frac{V}{a} \qquad n_b = \frac{V}{b} \qquad n_c = \frac{V}{c} \tag{26g}$$

where V is the distance light travels in 1 s in vacuum and a, b, and c are the semiaxes of the elliptic sections of the wave front. Values of n_a, n_b, and n_c are given for several crystals in Table 26C.

The distinction between positive and negative crystals is made according to to whether the angle α of Fig. 26J(a) is less or greater than 45°.

The angle α in Fig. 26J(a) can be calculated from the geometry of a circle and an ellipse, and is given by the relation,

$$\cos \alpha = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}} \tag{26h}$$

Table 26C PRINCIPAL INDICES OF REFRACTION FOR BIAXIAL CRYSTALS (FOR SODIUM LIGHT)

Crystal and formula	n_a	n_b	n_c	Angle α, deg
Negative crystals:				
Mica $[KH_2Al_3(SO_4)_3]$	1.5601	1.5936	1.5977	71.0
Aragonite $[CaO(CO)_2]$	1.5310	1.6820	1.6860	81.4
Lithargite (PbO)	2.5120	2.6100	2.7100	46.3
Stibnite$(Sb_2S_3)(\lambda 7620)$	3.1940	4.0460	4.3030	80.7
Positive crystals:				
Anhydrite $(CaSO_4)$	1.5690	1.5750	1.6130	22.1
Sulfur (S)	1.9500	2.0430	2.2400	37.3
Topaz $[(2AlO)FSiO_2]$	1.6190	1.6200	1.6270	20.8
Turquoise $(CuO_3 \cdot Al_2O_3 \cdot 2P_2O_5 \cdot 9H_2O)$	1.5200	1.5230	1.5300	33.3

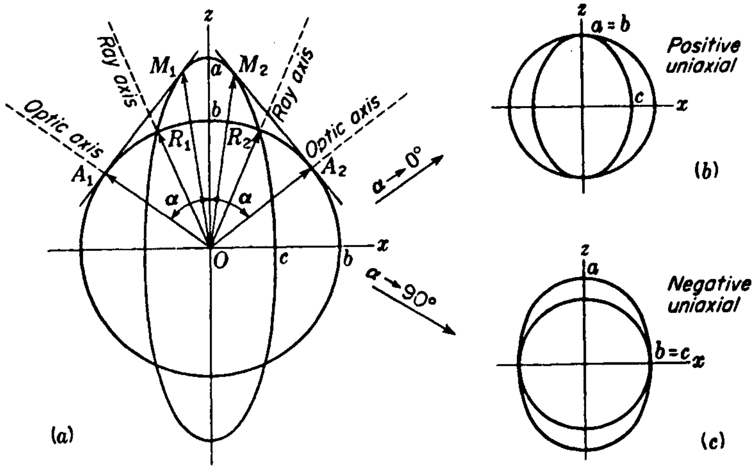

FIGURE 26J
Wave-surface diagram for (*a*) a biaxial crystal and (*b*) and (*c*) the limiting cases of uniaxial crystals.

It can be seen from the diagram that, as a approaches b, α approaches zero and the surface takes the form of a positive uniaxial crystal [Fig. 26J(*b*)]. When, on the other hand, $\alpha = 90°$, we have $b = c$ and the surface is that of a negative uniaxial crystal, as in (*c*) of the figure. In terms of the refractive indices these limiting cases are

$$n_a = n_b < n_c \qquad \textit{Positive uniaxial crystal having } n_O = n_a \textit{ or } n_b,\ n_E = n_c$$

$$n_a < n_b = n_c \qquad \textit{Negative uniaxial crystal having } n_O = n_b \textit{ or } n_c,\ n_E = n_a$$

It is to be noted in Fig. 26I that each coordinate plane contains one circular cross section of the wave surface. This means that one of the two rays refracted into a crystal along any of these planes will obey Snell's law. Prisms can therefore be cut from crystals in such a way as to make use of this fact in determining the principal indices of refraction.

One quadrant of the wave surface for a biaxial crystal is shown in Fig. 26K to illustrate the directions of the electric displacements **D**, that is, the vibrations in the wave fronts, and also to show the normal velocity surface (dashed lines). The outer sheet touches the inner one at only four points, where it forms "dimples." These are located at points such as R_2, where the surface is intersected by the ray axes. Along the x, y, and z axes the ray velocity is equal to the normal velocity. It will be seen that the vibrations in the wave surface wherever it has a circular section are perpendicular to the coordinate plane, for only under these conditions do they maintain a constant angle with the optic axes.

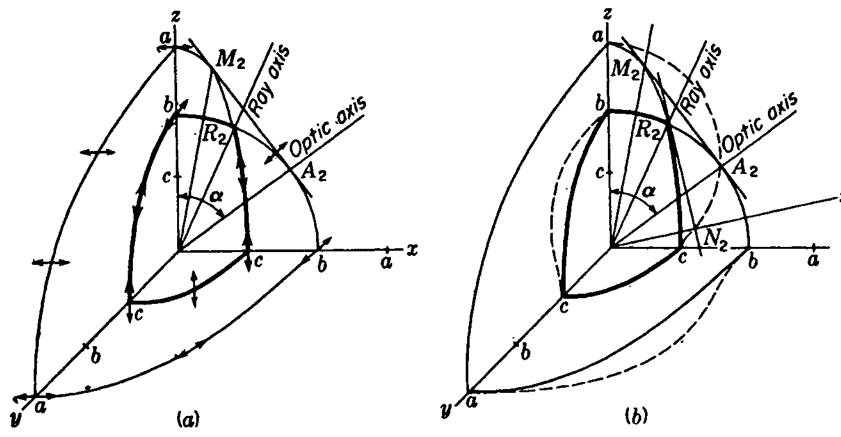

FIGURE 26K
Quadrant cross sections of wave surfaces for a biaxial crystal. Broken lines are normal-velocity surfaces. Arrows show the direction of the electric displacement.

26.7 INTERNAL CONICAL REFRACTION

The investigation of refraction in biaxial crystals follows the same lines as that given in the preceding sections for uniaxial crystals. To treat refraction in the xz plane, for example, we may apply Huygens' construction using secondary wavelets of the form shown in Fig. 26J. One finds in general two plane-polarized refracted rays, so that we have double refraction here also. Two special cases arise, however, in which the behavior of a biaxial crystal is different from the simpler uniaxial type. They correspond to the singular case where light is sent along the optic axis of a uniaxial crystal. One of these, *internal conical refraction*, is observed when a beam is directed along one of the *optic* axes inside the crystal. In the other, *external conical refraction*, it is sent along one of the *ray* axes.

Internal conical refraction comes about as follows: It has already been mentioned that the tangent plane A_2M_2 [Figs. 26J(a) and 26L(a)] makes contact with the three-dimensional wave surface in a circle of diameter A_2M_2. Suppose now that a plane-parallel plate is cut from a crystal so that its surfaces are perpendicular to an optic axis and that the crystal has the thickness OA_2 of Fig. 26L(a). Let a ray of unpolarized light be incident normally on the first surface at O. The perpendicular vibrations will travel along the optic axis OA_2 and pass through undeviated. The parallel vibrations will be propagated along OM_2, and will emerge after a second refraction traveling in the same direction as OA_2. Now the incident unpolarized ray contains vibrations in all planes through the ray (Sec. 24.2) and for each particular plane of vibration there is a different direction along which the wave will be propagated with the same normal velocity as along any other ray. In three dimensions these rays will form a cone of light in the crystal spreading out from O. Arriving simultaneously at the second surface A_2M_2, all these waves are refracted parallel to each other to form a circular cylinder. When this hollow beam of light is looked at end-on, the planes of vibration will be as shown in Fig. 26L(b).

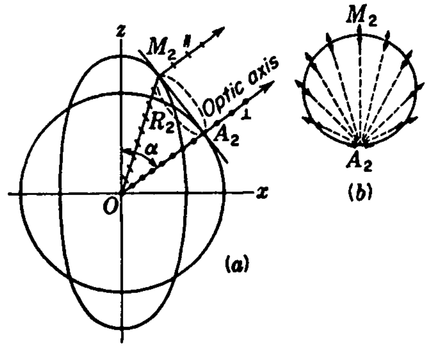

FIGURE 26L
(*a*) Geometry of internal conical refraction. (*b*) End view of internal conically refracted light, showing the directions of vibration.

Internal conical refraction was predicted by Sir William Hamilton and at his suggestion is reputed to have first been verified experimentally by Lloyd in 1833. The observations are usually made now with a parallel crystal plate as shown in Fig. 26M. A beam of light, confined to a narrow pencil by two movable pinholes S_1 and S_2, is incident at just such an angle that the light vibrating perpendicular to the plane of incidence is refracted along the optic axis. When the pinhole S_2 is moved around to vary the angle of incidence, there will be only two refracted rays until the correct direction for internal conical refraction is reached. When this happens, the light spreads out from the two spots near A_2 and M_2 into a ring.*

26.8 EXTERNAL CONICAL REFRACTION

External conical refraction in biaxial crystals deals with the refraction of an external hollow cone of light into a narrow pencil or ray of light inside the crystal (Figs. 26N and 26O). Suppose that a beam of monochromatic light is moving inside a crystal

* A photograph of internal conically refracted light is given in Max Born, "Optik," p. 240, J. Springer, Berlin, 1933.

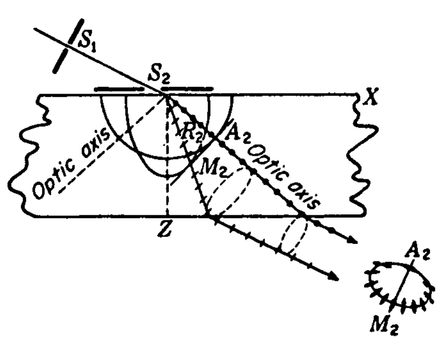

FIGURE 26M
Internal conical refraction in a biaxial crystal plate.

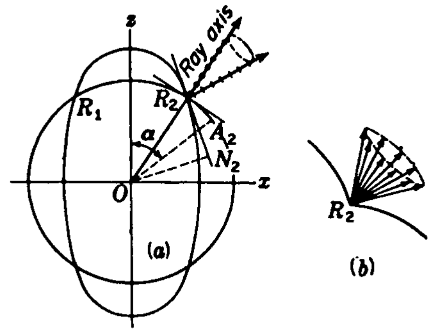

FIGURE 26N
Geometry of external conical refraction.

along the ray axis OR_2. From the diagram in Fig. 26N two tangents can be drawn at the intersection R_2, one to the ellipse and one to the circle.

In the three-dimensional wave surfaces the point R_2 is like a dimple, and there is an infinite number of wave fronts enveloping the obtuse cone. Corresponding to these wave fronts there will be an infinite number of wave normals, each with its own particular direction of vibration [Fig. 26N(b)], and these will form an acute-angled cone. When these wave fronts, the energy of each one of which travels along the ray axis, arrive at the crystal surface, they will emerge as a cone of rays, since each wave normal inside corresponds to a refracted ray outside. There is thus a cone of wave normals outside as well as inside. By the principle of the reversibility of light rays a hollow cone of polarized light rays from outside a crystal should unite to form one ray inside traveling along the single-ray axis.

Experimentally a solid cone of converging unpolarized light, somewhat larger than necessary, is made to fall on a crystal plate cut as shown in Fig. 26O. The ray axis is located by moving one of the pinhole apertures S_1 and S_2. From the incident light the crystal picks the hollow cone of rays vibrating in the proper planes that unite to form one ray. The various other rays travel different directions in the crystal and are stopped by the screen S_2. Upon refraction from the second crystal surface a hollow cone of polarized light is observed emerging from S_2. The cone shown in Fig. 26O is not the same as the one shown in Fig. 26N(b), but is the one produced by refraction of the latter.

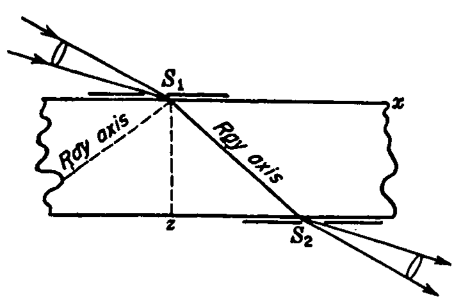

FIGURE 26O
Method of observing external conical refraction.

26.9 THEORY OF DOUBLE REFRACTION

Maxwell's equations for crystalline media have the same form as those given in Sec. 23.9 for transparent media in general, namely,

$$\begin{aligned} \frac{1}{c}\frac{\partial D_x}{\partial t} &= \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} & -\frac{1}{c}\frac{\partial H_x}{\partial t} &= \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} &= 0 & \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} &= 0 \end{aligned} \tag{26i}$$

Only in the case of an isotropic substance like glass, however, is it permissible to write for the electric displacement $\mathbf{D} = \varepsilon\mathbf{E}$, as was done in Sec. 23.9. In anisotropic crystals it is found that the measured values of the dielectric constant ε vary with the orientation of the optic axis or axes relative to the electric field $\mathbf{E}$. In the electron theory of dielectric media, the value of the dielectric constant depends on the *polarization* of the atoms under the influence of the electric field. This fact was mentioned in connection with our discussion of dispersion. The effect of the electric field is to produce a slight relative displacement of the positive and negative charges, so that the atom acquires an electric moment. Now the moment generated in a given atom depends on the electric field at that atom, which will be determined in part by the fields from other polarized atoms in its immediate neighborhood. If these other atoms are arranged in a particular way, it is clear that the polarization and the effective dielectric constant should depend on the orientation of the electric vector of the waves. In calcite, for example, the oxygen atoms in the CO_3 group are the most easily polarized, and they exert a strong influence on each other. Under this influence, they are more easily polarized by an electric field parallel to the plane of the group than by one perpendicular to it. At a result, we shall find that the refractive index should be greatest for light having its electric vector perpendicular to the trigonal axis.

The fact that ε varies with direction in these crystals can be shown by the electromagnetic theory to give rise to double refraction. The direction of $\mathbf{D}$ differs from that of $\mathbf{E}$ except in three singular directions, which are mutually perpendicular. The value of ε is a maximum along one of these axes, a minimum along another, and intermediate along the third. Designating them by x, y, and z, we therefore find that, for the three components of $\mathbf{D}$ in Maxwell's equations, we must now write

$$D_x = \varepsilon_x E_x \qquad D_y = \varepsilon_y E_y \qquad D_z = \varepsilon_z E_z \tag{26j}$$

When these values are substituted in Eqs. (26i), and the equation for plane electromagnetic waves derived,* it is found that for any direction of the wave front there are two velocities for vibrations of the vector $\mathbf{D}$ in two mutually perpendicular directions, and this is the fundamental aspect of double refraction.

The most concise way of presenting the results of the electromagnetic theory

* See, for example, P. Drude, "Theory of Optics," English edition, pp. 314–317, Longmans, Green & Co., Inc., New York, 1922.

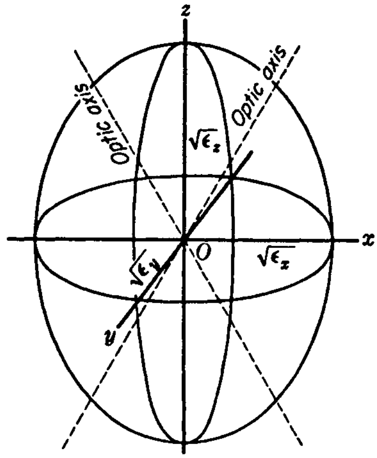

FIGURE 26P
Dielectric ellipsoid for a biaxial crystal.

is by the use of what we may call the *dielectric ellipsoid*. This is an ellipsoid described by the equation

$$\frac{x^2}{\varepsilon_x} + \frac{y^2}{\varepsilon_y} + \frac{z^2}{\varepsilon_z} = 1 \qquad \text{(26k)}$$

in which ε_x, ε_y, and ε_z are the *principal dielectric constants* of Eqs. (26j). The semiaxes of the ellipsoid are $\sqrt{\varepsilon_x}$, $\sqrt{\varepsilon_y}$, and $\sqrt{\varepsilon_z}$, as shown in Fig. 26P, where we have taken $\varepsilon_x < \varepsilon_y < \varepsilon_z$. From the ellipsoid we may obtain the two velocities and the corresponding directions of vibration for a wave traveling in any arbitrary direction through the crystal, as explained below. This mode of representation was first given by Fresnel in terms of the elastic-solid theory of light. Since, in the older theory, the velocity was dependent on the elasticity and density of the ether, Fresnel's ellipsoid could be either an "ellipsoid of elasticity" or an "ellipsoid of inertia." When it is replaced by a dielectric ellipsoid, Fresnel's results can be translated directly into the terms of the electromagnetic theory.

Suppose now that ordinary light waves vibrating in all planes are moving through the point O in the crystal in every direction and that we wish to determine the double wave surfaces presented in the preceding sections. In Eq. (23p), the velocity of light was given by

$$v = \frac{c}{\sqrt{\varepsilon}} \qquad \text{(26l)}$$

where c is the velocity in vacuo. We therefore have the relations

$$v_a = \frac{c}{\sqrt{\varepsilon_x}} \qquad v_b = \frac{c}{\sqrt{\varepsilon_y}} \qquad v_c = \frac{c}{\sqrt{\varepsilon_z}}$$

$$n_a = \sqrt{\varepsilon_x} \qquad n_b = \sqrt{\varepsilon_y} \qquad n_c = \sqrt{\varepsilon_z}$$

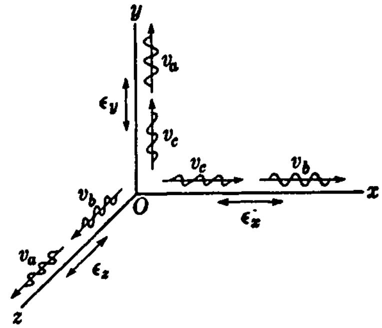

FIGURE 26Q
Correlation of the velocities and directions of vibration in the waves with the directions of the three principal dielectric constants.

where $v_a > v_b > v_c$. Now v_a represents the velocity of waves traveling perpendicular to the x axis with their electric displacements parallel to x. Their velocity is therefore determined by ε_x. The application of this statement to the other directions of vibration and velocities of propagation along the three coordinate axes may be seen by inspection of Fig. 26Q.

Let us now see how the two velocities in any arbitrary direction may be determined by the use of the dielectric ellipsoid. First we note that the velocities along any one of the coordinate axes are inversely proportional to the major and minor axes of the elliptical section of the ellipsoid made by the coordinate plane that is perpendicular to that axis. In the same way, for any other direction of propagation, we pass a plane through O parallel to the plane of the wave. This will cut the ellipsoid in an ellipse with major and minor axes OA and OB, Fig. 26R(a). On the wave normal the distances OM and ON are measured off inversely proportional to OA and OB.

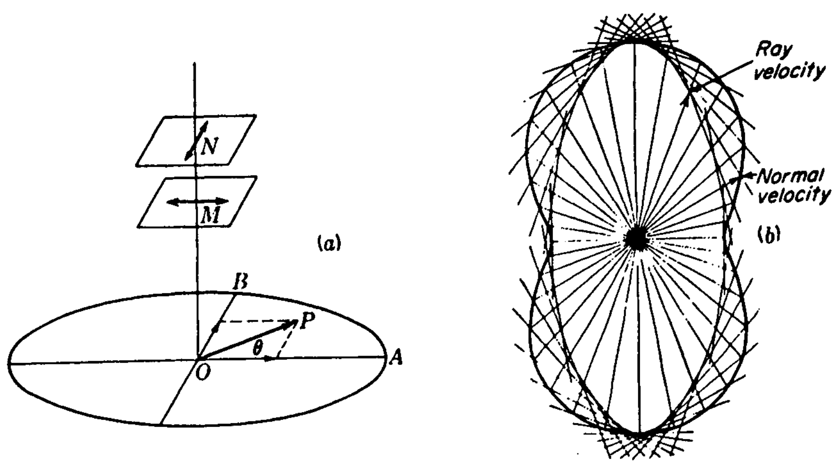

FIGURE 26R
The construction of a normal-velocity surface.

The planes M and N parallel to the initial plane represent a later position of the waves vibrating parallel to the two axes of the ellipse. If we consider a single vibration in plane AOB making an angle θ with OA, the electric vector OP may be resolved into two components $OP \cos \theta$ and $OP \sin \theta$. These components along the major and minor axes travel with the two different velocities. If now the plane AOB is rotated about O in every possible direction the points M and N will trace out the normal-velocity surfaces (dotted lines) shown in Fig. 26K(*b*).

For every ellipsoid with three different axes there are two planes only for which the cross sections are circles. For these two planes, OA and OB in Fig. 26R(*a*) will be equal and the planes M and N will coincide. The directions of the normal for these two circular cross sections of the dielectric ellipsoid give the optic axes of the crystal, i.e., the directions of equal *normal* velocity for all planes of vibration. The envelope of all plane waves at the instant they reach the normal-velocity surface is the wave surface previously described in Sec. 26.6. This envelopment, giving a surface of elliptical section, is illustrated in Fig. 26R(*b*).

The optical properties of doubly refractive crystals are completely determined when one knows the values of the three principal refractive indices and the directions of two of the principal axes. As was mentioned, these can be measured by cutting the crystal into prisms of different orientation. There exist, however, more powerful and more convenient methods based on the interference effects resulting from the difference in velocity of the two polarized components, and these will be discussed in the next chapter.

PROBLEMS

26.1 A light ray strikes the surface of a crystal of ice at grazing incidence in a plane perpendicular to the optic axis. The crystal has been cut so that its axis lies parallel to the surface. Find the separation in millimeters of the O and E rays at the opposite face of the crystal, which is a plane-parallel slab 4.20 mm thick. Assume $n_O = 1.3090$ and $n_E = 1.3104$ for sodium light. *Ans.* 0.01271 mm

26.2 Find by graphical construction how thick a natural calcite crystal would need to be in order for a ray of sodium light incident normally on one cleavage face to emerge from the opposite face as two rays separated by a linear distance of 2.50 mm. In a principal section of calcite, the optic axis can be assumed to make an angle of 45° with the normal.

26.3 A ray of unpolarized light falls on a calcite crystal, the optic axis of which is parallel to the surface. The angle of incidence is 32°, and the plane of incidence coincides with the principal section of the crystal. Find the angles of refraction of the O and E rays for the green mercury line (see Table 26A and footnote Sec. 26.3).

26.4 A 50° prism is made of ammonium phosphate for which $n_O = 1.5250$ and $n_E = 1.4790$. If the prism is cut with its optic axis parallel to its refracting edge, calculate (*a*) the angles of minimum deviation and (*b*) their difference.
Ans. (*a*) $\delta_O = 30.26°$, $\delta_E = 27.37°$, (*b*) 2.89°

26.5 Draw to scale the two cross sections of the wave surface of Rutile (TiO_2) made by planes (*a*) parallel to the optic axis and (*b*) perpendicular to the axis. Indicate the directions of the vibrations on each diagram. (*c*) Is rutile a positive or negative crystal? Assume the light is for the Fraunhofer F line, $\lambda = 4861$ Å.

26.6 The angle 2α between the optic axes of a biaxial crystal is given by Eq. (26g). The principal indices of refraction for two unidentified crystals are measured and found to be (*a*) for the first, $n_a = 1.6842$, $n_b = 1.6935$, and $n_c = 1.7126$ and (*b*) for the second, $n_a = 2.1547$, $n_b = 2.3282$, and $n_c = 2.4034$. Find the angle α for both crystals, and determine whether they are positive or negative.

Ans. (*a*) 35.24° positive, (*b*) 58.77° negative

26.7 Draw to scale cross sections in the three coordinate planes of the wave surfaces of a biaxial crystal of sulfur. See Table 26C for refractive indices.

26.8 Construct one quadrant of the xz section of the elliptical wave surface for stibnite. From this graphically construct the corresponding normal velocity surface in this same plane [see Fig. 26R(*b*)]. Show the optic axis.

26.9 A crystal of stibnite is cut into a 20° prism with its refracting edge perpendicular to the plane containing the optic axes. The angle of minimum deviation is measured for a ray of sodium light with its vibrations parallel to the refracting edge. What value would be expected according to the refractive indices given in Table 26C?

Ans. $\delta = 69.3°$

26.10 The axis of *single ray velocity* in a biaxial crystal makes an angle β with the z axis, the cosine of which is a/b times as large as the value of $\cos\alpha$. Find the apex angle of the cone of internal conical refraction in a crystal of stibnite, using the refractive indices given in Table 26C.