Exercises

ind $\frac{dy}{dx}$

$= x^3\sin(2x^2+3)$ ② $y=\sin(\cos x)$ ③ $y=\sin(\tan 3x)$

$= \sin\sqrt{x}$ ⑤ $y=\sqrt{\sin x}$ ⑥ $y=\sin^3(9x+1)$ ⑦ $y=[1+\sin^3(x^5)]^{12}$

$= \cos(\cos x)$ ⑨ $y=4\cos x^9$ ⑩ $y=\cos^3(\sin 2x)$ ⑪ $y=\tan^2 x$

$= [\cos^2(1-x)+\sqrt{x+3}]^5$ ⑬ $y=\sin 3x\cos 3x$ ⑭ $y=x^5\sec(\frac{1}{x})$

$\cot(xy)+xy=0$ ⑯ $x=\csc(x-y)+\sec(x+y)$

$y=\frac{\sin x}{\sec(3x+1)}$ ⑱ $y=\frac{1+\csc(x^2)}{1-\cot(x^2)}$ ⑲ $y=[x^4-\sec(4x^2-2)]^{-4}$

ind an equation for the tangent line to the graph of

$= \sin 2x+\cos 2x$ at $x_0=\frac{\pi}{2}$ ⓑ $y=\sec^3(\frac{\pi}{2}-x)$ at $x=\frac{-\pi}{2}$

nverse Trigonometric Functions (الدوال المثلثية العكسية)

finition: A function $f: A \to B$ is said to be one-to-one

1) if $\forall x_1, x_2 \in A$, $f(x_1)=f(x_2) \Rightarrow x_1=x_2$

finition: A function $f: A \to B$ is said to be onto if

$\forall y \in B, \exists x \in A$ such that $f(x)=y$

: $f: A \to B$ is said to be onto if $f(A)=B$

amples

Let $f: R \to R \ni f(x)=2x+3$

f is (one-to-one) since

$\forall x_1, x_2 \in R$ and $f(x_1)=f(x_2) \Rightarrow 2x_1+3=2x_2+3 \Rightarrow 2x_1=2x_2$

$\Rightarrow x_1=x_2 \Rightarrow f$ is (1-1)

f is onto since

$B=R$ and $R_f=R \longrightarrow B=R_f=R \longrightarrow f$ is onto

(2) Let $f: R \to R^+ \cup \{0\} \ni f(x)=x^2$

f is not (1-1) since $f(2)=f(-2)=4$, but $2 \neq -2$

f is onto since

$B=R^+ \cup \{0\}$ and $R_f=R^+ \cup \{0\} \longrightarrow B=R_f \Longrightarrow f$ is onto

* إذا كانت الدالة $f: A \to B$ متباينة وشاملة فإن لكل
y في مداها B يوجد عنصر وحيد في منطلقها A وعليه
نعرف دالة أخرى يكون منطلقها B ومداها A وهي معكوس
الدالة f والتي سنرمز لها بالرمز f^{-1} حيث أن

$f^{-1}: B \to A$ بعبارة أخرى:

$$y=f(x) \Longleftrightarrow x=f^{-1}(y)$$

لاحظ أن $f^{-1}(y)$ لا تساوي $\frac{1}{f(y)}$

Example:

$y=f(x)=2x+3$ is (1-1) and onto function, then f has an inverse function $f^{-1}(y)=\frac{y-3}{2}$

الدوال المثلثية بصورة عامة ليس لها معكوس (لأنها غير متباينة)
أما إذا جعلنا مجالها ضمن فترة معينة تجعل تلك الدوال
متباينة وشاملة وبذلك نستطيع إيجاد معكوسها . والآن
سوف ندرس معكوس كل دالة من الدوال المثلثية الآتية وكما
يلي:

~~e inverse sine function~~ (معكوس دالة الجيب)

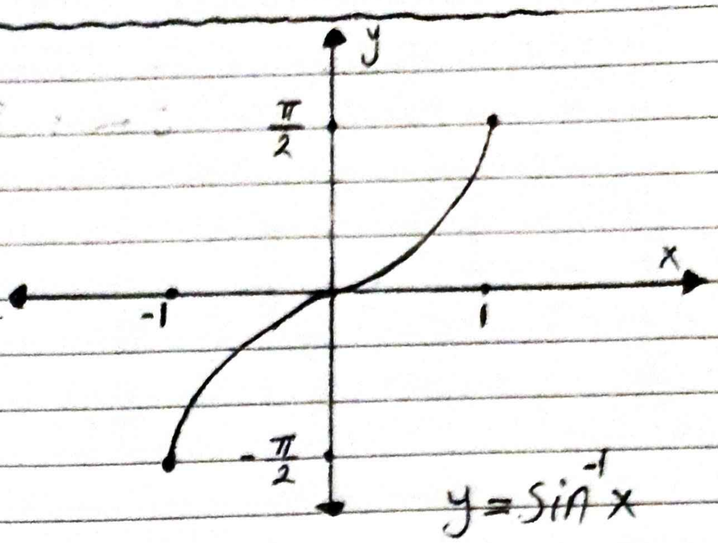
$= \sin^{-1} x \longleftrightarrow \sin y = x$

$[-1,1]$ ② $R_f = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$n(\sin^{-1} x) = x$ if $x \in [-1,1]$

$^{-1}(\sin x) = x$ if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$in^{-1}(-x) = -\sin^{-1} x$



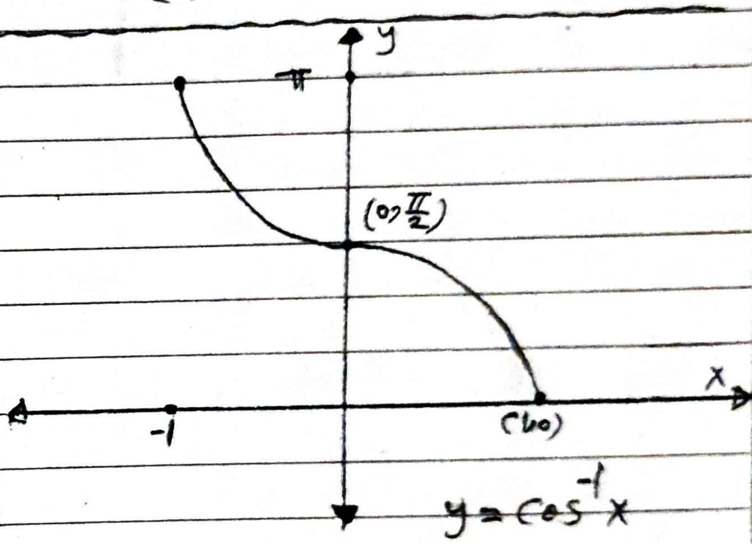
~~e inverse cosine function~~ (معكوس دالة الجيب تمام)

$) = \cos^{-1} x \longleftrightarrow \cos y = x$

$= [-1,1]$ ② $R_f = [0, \pi]$

$s(\cos^{-1} x) = x$ if $x \in [-1,1]$

$s^{-1}(\cos x) = x$ if $x \in [0, \pi]$



~~The inverse tangent function~~ (معكوس دالة الظل)

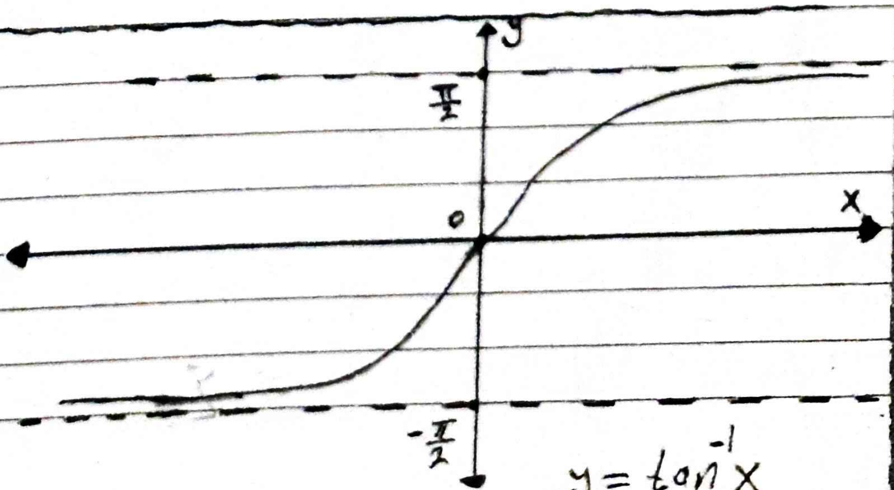
$= \tan^{-1} x \longleftrightarrow \tan y = x$

$\| R$ ② $R_f = (-\frac{\pi}{2}, \frac{\pi}{2})$

$n^{-1}(\tan x) = x$ if $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$n(\tan^{-1} x) = x$ if $x \in R$

$n^{-1}(-x) = -\tan^{-1}(x)$



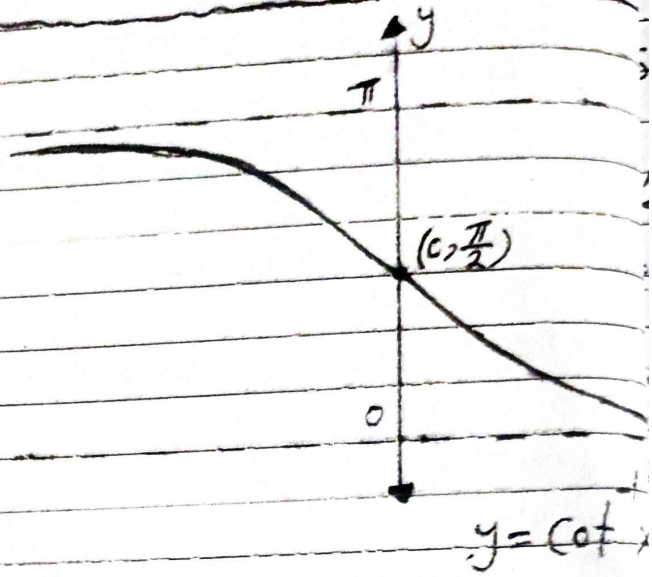
~~4) The inverse cotangent function~~ ~~…دالة الظل تمام)~~

$y = \cot^{-1} x \longleftrightarrow \cot y = x$

1) $D_f = R$ 2) $R_f = (0, \pi)$

3) $\cot(\cot^{-1} x) = x$ if $x \in R$

4) $\cot^{-1}(\cot x) = x$ if $x \in (0, \pi)$



5) The inverse secant function …وس دالة القاطع)

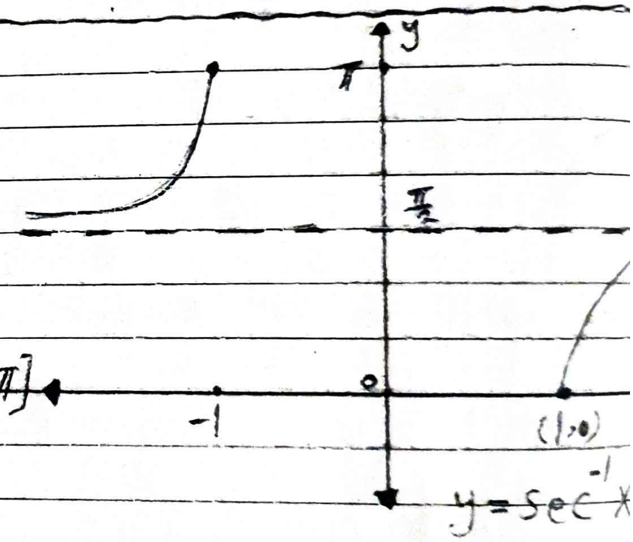
$y = \sec^{-1} x \longleftrightarrow \sec y = x$

1) $D_f = |x| \geq 1$

2) $R_f = [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

3) $\sec^{-1}(\sec x) = x$ if $x \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

4) $\sec(\sec^{-1} x) = x$ if $|x| \geq 1$

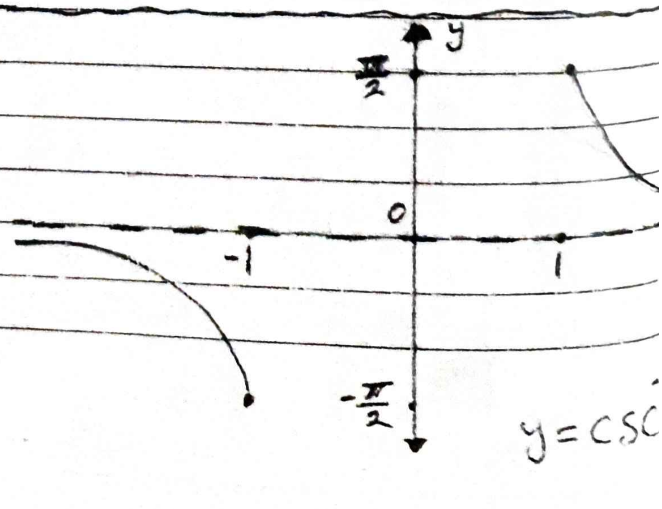


6) The inverse cosecant function …كوس دالة القاطع تمام)

$y = \csc^{-1} x \longleftrightarrow \csc y = x$

1) $D_f = |x| \geq 1$

2) $R_f = [\frac{-\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$



$csc^{-1}(\csc x) = x$ if $x \in [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}]$

$\csc(\csc^{-1} x) = x$ if $|x| \geq 1$

rks: ① $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ ② $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$

rove that ① $\sec^{-1}x = \cos^{-1}(\frac{1}{x})$ ② $\csc^{-1}x = \sin^{-1}(\frac{1}{x})$ ch.

et $y = \sec^{-1}x \longrightarrow \sec y = x \longrightarrow \frac{1}{\cos y} = x \longrightarrow$

$\cos y = \frac{1}{x} \longrightarrow y = \cos^{-1}(\frac{1}{x}) \longrightarrow \sec^{-1}x = \cos^{-1}(\frac{1}{x})$

Find the value of

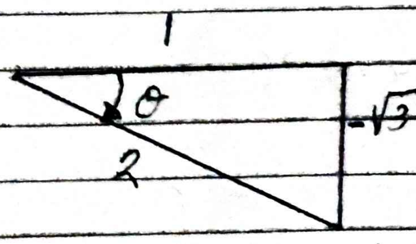
$n^{-1}(\frac{1}{2})$ ② $\sin^{-1}(-\frac{1}{\sqrt{2}})$ ③ $\sin^{-1}(-1)$ ④ $\cos^{-1}(-1)$ ⑤ $\tan^{-1}(-1)$

$t^{-1}(1)$ ⑦ $\sec^{-1}(1)$ ⑧ $\csc^{-1}(1)$ ⑨ $\tan^{-1}(1)$ ⑩ $\cot^{-1}(-1)$

et $y = \sin^{-1}(-\frac{1}{\sqrt{2}}) \longrightarrow \sin y = -\frac{1}{\sqrt{2}} \longrightarrow y = -\frac{\pi}{4}$

$\sin^{-1}(-\frac{1}{\sqrt{2}}) = -\frac{\pi}{4}$

Given that $\theta = \sin^{-1}(-\frac{\sqrt{3}}{2})$, find the values of $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$



$\theta = \sin^{-1}(-\frac{\sqrt{3}}{2}) \longrightarrow \sin\theta = -\frac{\sqrt{3}}{2}$

$\cos\theta = \frac{1}{2}$, $\tan\theta = -\sqrt{3}$, $\cot\theta = -\frac{1}{\sqrt{3}}$, $\sec\theta = 2$ and

$\csc\theta = -\frac{2}{\sqrt{3}}$

23) Find the value of

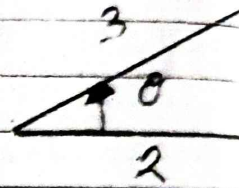
① $\sec(\sin^{-1}(\frac{-2}{3}))$ ② $\sin(2\cos^{-1}(\frac{3}{5}))$ ③ $\tan(2\cos^{-1}(\frac{4}{5}))$ ④ $\sin(co$

⑤ $\cos(2\sin^{-1}(\frac{-5}{13}))$ ⑥ $\tan(2\sec^{-1}(\frac{3}{2}))$ ⑦ $\sin[\sin^{-1}(\frac{2}{3}) + \cos^{-1}($

⑧ $\cos[\sin^{-1}(\frac{1}{3}) - \tan^{-1}(\frac{1}{2})]$ ⑨ $\tan[\tan^{-1}(\frac{3}{4}) - \sin^{-1}(\frac{1}{2})]$ ⑩ $\tan(si$

Sol

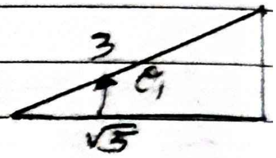
⑥ Let $\theta = \sec^{-1}(\frac{3}{2}) \longrightarrow \sec\theta = \frac{3}{2}$



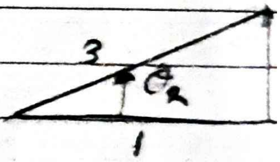
$\therefore \tan(2\sec^{-1}(\frac{3}{2})) = \tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\cdot\frac{\sqrt{5}}{2}}{1-(\frac{\sqrt{5}}{2})^2} = \frac{\sqrt{5}}{1-\frac{5}{4}} = \frac{\sqrt{5}}{\frac{-1}{4}} = -4\sqrt{5} \longrightarrow \tan(2\sec^{-1}(\frac{3}{2})) = -$

⑦ Let $\theta_1 = \sin^{-1}(\frac{2}{3}) \longrightarrow \sin\theta_1 = \frac{2}{3}$



And $\theta_2 = \cos^{-1}(\frac{1}{3}) \longrightarrow \cos\theta_2 = \frac{1}{3}$



$\therefore \sin[\sin^{-1}(\frac{2}{3}) + \cos^{-1}(\frac{1}{3})] = \sin(\theta_1 + \theta_2)$

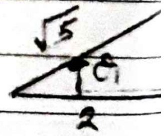
$= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \frac{2}{3}\cdot\frac{1}{3} + \frac{\sqrt{5}}{3}\cdot\frac{\sqrt{8}}{3} = \frac{2}{9} + \frac{\sqrt{}}{9}$

$= \frac{2+\sqrt{40}}{9}$

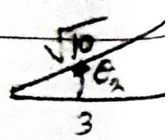
24) prove that ① $\cos^{-1}(\frac{3}{\sqrt{10}}) + \cos^{-1}(\frac{2}{\sqrt{5}}) = \frac{\pi}{4}$

② $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3}) = \frac{\pi}{4}$ ③ $2\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{7}) = \frac{\pi}{4}$

Sol ② let $\theta_1 = \tan^{-1}(\frac{1}{2}) \longrightarrow \tan\theta_1 = \frac{1}{2}$



And let $\theta_2 = \tan^{-1}(\frac{1}{3}) \longrightarrow \tan\theta_2 = \frac{1}{3}$



نأخذ ظل الطرفين الايسر وكذلك : [illegible]

$\tan\left[\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)\right] = \tan(\theta_1 + \theta_2) = \dfrac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$

$= \dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = \dfrac{\frac{3+2}{6}}{1 - \frac{1}{6}} = \dfrac{\frac{5}{6}}{\frac{5}{6}} = 1 = \tan\left(\frac{\pi}{4}\right)$

$\tan(\theta_1 + \theta_2) = \tan\left(\frac{\pi}{4}\right) \longrightarrow \tan^{-1}(\tan(\theta_1 + \theta_2)) = \tan^{-1}\left(\tan\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \theta_1 + \theta_2 = \frac{\pi}{4} \Longrightarrow \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Find the value of x, if

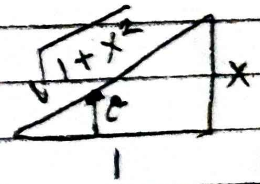
$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}(x) = \tan^{-1}\left(\frac{4}{x}\right)$ ② $\tan\left(\frac{\pi}{4} + \tan^{-1}(x)\right) = \tan^{-1}\sqrt{3}$

$\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$

Simplify the function

$\cos(\sin^{-1}x)$ ② $\sec^2(\tan^{-1}x)$

② Let $\theta = \tan^{-1}x \longrightarrow \tan\theta = x$



$\sec^2(\tan^{-1}x) = \sec^2\theta = (\sec\theta)^2$

$= \left(\sqrt{1+x^2}\right)^2 = 1 + x^2$

Derivatives of the inverse Trigonometric functions
مشتقة الدوال المثلثية العكسية

① If $y(x) = \sin^{-1}(x) \rightarrow y' = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

proof

$\because y = \sin^{-1}(x) \rightarrow \sin y = x \rightarrow \cos y \frac{dy}{dx} = 1 \rightarrow$

$\rightarrow \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

② If $y = \cos^{-1}(x) \rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$ ch.

③ If $y = \tan^{-1}(x) \rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$

proof

$\because y = \tan^{-1}(x) \rightarrow \tan y = x \rightarrow \sec^2 y \frac{dy}{dx} = 1 \rightarrow$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2} \rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$

④ If $y = \cot^{-1}(x) \rightarrow \frac{dy}{dx} = \frac{-1}{1+x^2}$ ch.

⑤ If $y = \sec^{-1}(x) \rightarrow \frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ ch.

⑥ If $y = \csc^{-1}(x) \rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-1}}$ ch.

بصورة عامة اذا كانت $u = g(x)$ حيث إن $g(x)$ دالة قابلة للاشتقاق بالنسبة لـ x فإن بموجب قاعدة السلسلة مشتقة الدوال الدائرية تصبح بالشكل الآتي :

$y(x) = \sin^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-u^2}} \dfrac{du}{dx}$

$y(x) = \cos^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-u^2}} \dfrac{du}{dx}$

$y(x) = \tan^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{1}{1+u^2} \dfrac{du}{dx}$

$y(x) = \cot^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{-1}{1+u^2} \dfrac{du}{dx}$

$y(x) = \sec^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{1}{u\sqrt{u^2-1}} \dfrac{du}{dx}$

$y(x) = \csc^{-1} u \longrightarrow \dfrac{dy}{dx} = \dfrac{-1}{u\sqrt{u^2-1}} \dfrac{du}{dx}$

Exs: Find $\dfrac{dy}{dx}$

$= \sin^{-1}(x^2) \longrightarrow \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-(x^2)^2}} \cdot \dfrac{d}{dx}[x^2]$

$\dfrac{1}{\sqrt{1-x^4}} \cdot 2x \longrightarrow y' = \dfrac{2x}{\sqrt{1-x^4}}$

$= \cos^{-1}\sqrt{x} \longrightarrow y' = \dfrac{-1}{\sqrt{1-(\sqrt{x})^2}} \cdot \dfrac{d}{dx}[\sqrt{x}]$

$\dfrac{-1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}} \longrightarrow y' = \dfrac{-1}{2\sqrt{x-x^2}}$

$= x\sin^{-1}(2x) \longrightarrow y' = x \cdot \dfrac{1}{\sqrt{1-(2x)^2}} \cdot 2 + \sin^{-1}(2x)$

$$\therefore y' = \frac{2x}{\sqrt{1-4x^2}} + \sin^{-1}(2x)$$

④ $y = x(\sin^{-1}x)^2 - 2x + 2\sqrt{1-x^2}\,\sin^{-1}x$

$$\therefore y' = x \cdot 2(\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}} + (\sin^{-1}x)^2 \cdot 1 - 2 +$$

$$2\cancel{\sqrt{1-x^2}} \cdot \frac{1}{\cancel{\sqrt{1-x^2}}} + \sin^{-1}x \cdot \frac{\cancel{2}}{\cancel{2}}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

$$y' = \cancel{\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}} + (\sin^{-1}x)^2 - \cancel{2} + \cancel{2} - \cancel{\frac{2x\sin^{-1}x}{\sqrt{1-x^2}}} = (\sin$$

⑤ $y = 3\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{9-x^2}$

$$y' = \cancel{3}\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \cdot \frac{1}{\cancel{3}} + \frac{1}{\cancel{2}}(9-x^2)^{-\frac{1}{2}} \cdot (-\cancel{2}x)$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{9}}} - \frac{x}{\sqrt{9-x^2}} = \frac{1}{\sqrt{\frac{9-x^2}{9}}} - \frac{x}{\sqrt{9-x^2}} = \frac{3}{\sqrt{9-x^2}} -$$

$$= \frac{3-x}{\sqrt{9-x^2}} \longrightarrow y' = \frac{3-x}{\sqrt{9-x^2}}$$

⑥ $y = \cot^{-1}\left(\frac{2}{x}\right) + \tan^{-1}\left(\frac{x}{2}\right)$

$$y' = \frac{-1}{1+\left(\frac{2}{x}\right)^2} \cdot \left(\frac{-2}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2} \cdot \frac{1}{2}$$

$$y' = \frac{-1}{1+\frac{4}{x^2}} \cdot \left(\frac{-2}{x^2}\right) + \frac{1}{1+\frac{x^2}{4}} \cdot \frac{1}{2} = \frac{-1}{\frac{x^2+4}{x^2}} \cdot \left(\frac{-2}{x^2}\right) + \frac{1}{\frac{4+x}{4}}$$

$$=\frac{2\cancel{x^2}}{\cancel{x^2}(x^2+4)}+\frac{\cancel{4}\,2}{\cancel{2}(4+x^2)}=\frac{2}{x^2+4}+\frac{2}{x^2+4}=\frac{4}{x^2+4}$$

$$y=\sec^{-1}\sqrt{x^2+4}$$

$$y'=\frac{1}{\sqrt{x^2+4}\sqrt{(\sqrt{x^2+4})^2-1}}\cdot\frac{1}{\cancel{2}}(x^2+4)^{-1/2}\cdot\cancel{2}x$$

$$=\frac{1}{\sqrt{x^2+4}\sqrt{x^2+4-1}}\cdot\frac{x}{\sqrt{x^2+4}}=\frac{x}{(x^2+4)\sqrt{x^2+3}}$$

$$y=[1+x\csc^{-1}x]^{10}$$

$$y'=10[1+x\csc^{-1}x]^9\cdot\frac{d}{dx}[1+x\csc^{-1}x]$$

$$=10[1+x\csc^{-1}x]^9\cdot\left[\cancel{x}\cdot\frac{-1}{\cancel{x}\sqrt{x^2-1}}+\csc^{-1}x\right]$$

$$=10[1+x\csc^{-1}x]^9\cdot\left[\frac{-1}{\sqrt{x^2-1}}+\csc^{-1}x\right]$$

Exercise

Find $\frac{dy}{dx}$

① $y=\sin^{-1}\left(\frac{x}{3}\right)$ ② $y=\cos^{-1}(2x+1)$ ③ $y=\tan^{-1}(x^2)$ ④ $y=\sin^{-1}x+\cos^{-1}x$

⑤ $y=\cot^{-1}(\sqrt{x})$ ⑥ $y=\sec^{-1}(x^7)$ ⑦ $y=x\cos^{-1}(2x)-\frac{1}{2}\sqrt{1-4x^2}$

⑧ $y=x^3\cot^{-1}\left(\frac{x}{3}\right)$ ⑨ $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ ⑩ $y=x\csc^{-1}\left(\frac{1}{x}\right)$ ⑪ $y=(\tan x)^{-1}$

⑫ $y=\sin^{-1}\left(\frac{1}{x}\right)$ ⑬ $y=\cos^{-1}(\cos x)$ ⑭ $y=\tan^{-1}\left(\frac{1-x}{1+x}\right)$ ⑮ $y=x^2(\sin^{-1}x)^3$