The definite integral (التكامل المحدد)

Definition of Area: Suppose that f is a continuous function that is nonnegative on an interval $[a,b]$, and let R denote the region that is bounded below by the x-axis, bounded on the sides by the vertical lines $x=a$ and $x=b$, and bounded above by the curve $y=f(x)$. To define the area of the region R, we will use the rectangle method as follows:-

① Divide the interval $[a,b]$ into n $(n \in Z^+)$ equal subintervals.

② over each subinterval construct a rectangle whose height is the value of f at any point in the subinterval.

③ The union of these rectangles forms a region R_n whose area can be regarded as an approximation to the area A of the region R.

④ Repeat the process using more and more subdivisions.

⑤ Define the area of R to be the limit of the areas of the approximating regions R_n as n is made larger and larger without bound, that is

$$A = \text{area}(R) = \lim_{n \to \infty} [\text{area}(R_n)]$$

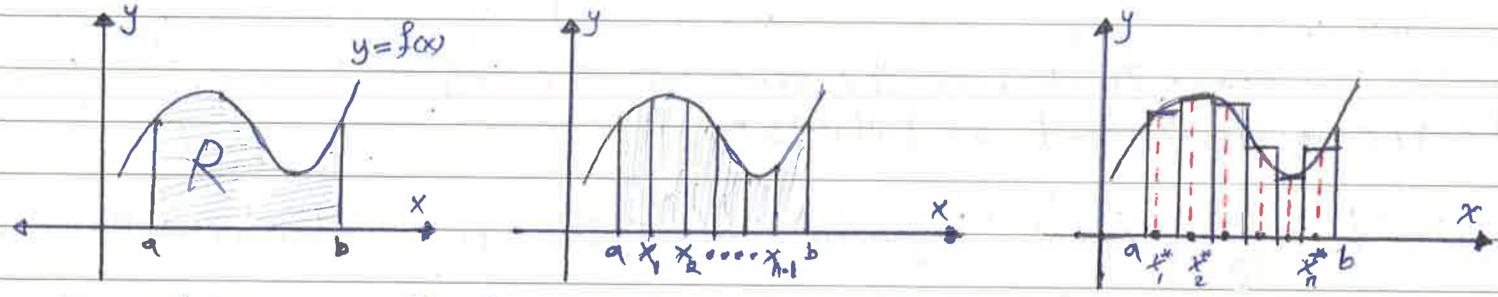
For example, suppose that we divide the interval $[a,b]$ into n subintervals by inserting $(n-1)$ equally spaced points between a and b, say

$$x_1, x_2, \text{------}, x_{n-1}$$

Each of these subintervals has width $\Delta x = \frac{b-a}{n}$

In each subinterval we need to choose an x-value at which to evaluate the function f to determine the height of a rectangle over the interval. If we denote those x-values by:-

$$x_1^*, x_2^*, \ldots\ldots, x_n^*$$



Then the areas of the rectangles constructed over these intervals will be

$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \ldots\ldots, f(x_n^*)\Delta x$$

And the total area of the region R_n will be

$$\text{area}(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

As $n \to \infty$, we get:-

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

From above we get the following definition

Definition (Area under a curve): If the function f is continuous on $[a,b]$ and if $f(x) \geq 0$ for all x in $[a,b]$, then the area under the curve $y = f(x)$ over the interval $[a,b]$ is defined by:-

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \longrightarrow (1)$$

Remark: In (1) the values of x_k^* $(k=1,2,\cdots,n)$ may be chosen in many different ways as follows:-

(1) x_k^* is the left endpoint of each subinterval

(2) x_k^* is the right endpoint of each subinterval

(3) x_k^* is the midpoint of each subinterval

If the subinterval $[a,b]$ is divided by $x_1, x_2, x_3, \cdots, x_{n-1}$ into n equal parts each of length $\Delta x = \frac{b-a}{n}$, and if we let $x_0 = a$ and $x_n = b$, then $x_k = a + k\Delta x$ for $k = 0,1,2,\cdots,n$

thus,

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \qquad \text{(Left endpoint)} \longrightarrow (2)$$

$$x_k^* = x_k = a + k\Delta x \qquad \text{(Right endpoint)} \longrightarrow (3)$$

$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + (k - \tfrac{1}{2})\Delta x \qquad \text{(Midpoint)} \longrightarrow (4)$$

Sigma notation

(1) $\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$

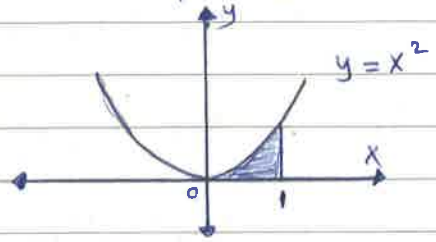
(2) $\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

(3) $\sum_{k=1}^{n} 1 = 1+1+1+\cdots+1 = n$

(4) $\sum_{k=1}^{5} 2 = 2+2+2+2+2 = 10$

Example: Use above definition with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x)=x^2$ and the interval $[0,1]$.

Solution: We have



$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

And from ③

$$x_k^* = a + k\,\Delta x = 0 + k \cdot \frac{1}{n} = \frac{k}{n}$$

So that

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x = \sum_{k=1}^{n} (x_k^*)^2\,\Delta x = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \frac{1}{n^3}\sum_{k=1}^{n} k^2$$

$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}$$

Therefore,

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

Remark: If f is continuous and assumes both positive and negative values on $[a,b]$, then the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x \quad \text{------} (**)$$

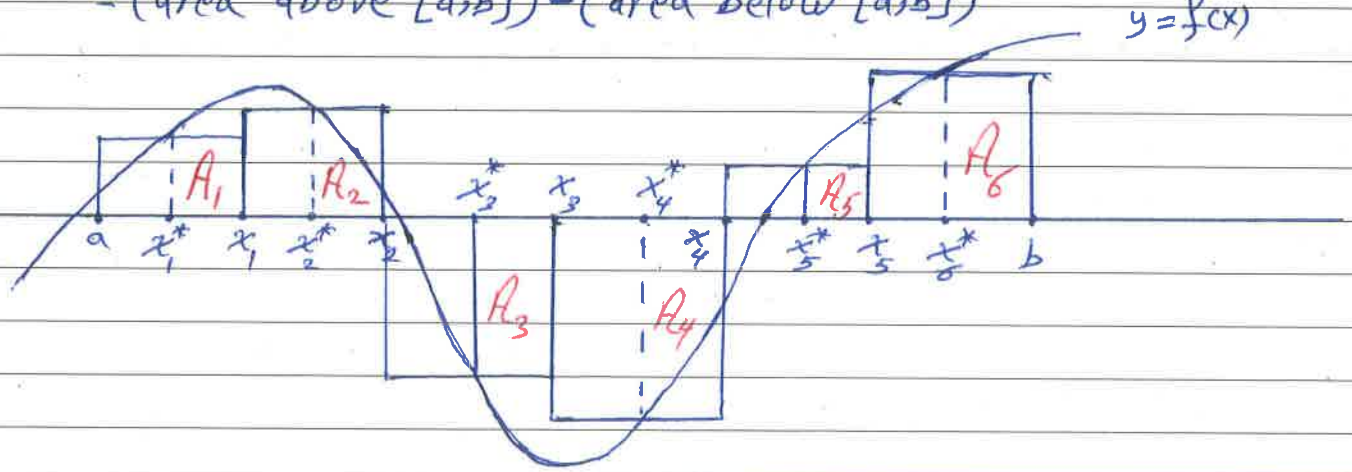
no longer represents the area between the curve $y=f(x)$ and the interval $[a,b]$ on the x-axis; rather, it represents a difference of areas — the area of the region that is above the interval $[a,b]$ and below the curve $y=f(x)$ minus the area of the region that is below the interval $[a,b]$ and above the curve $y=f(x)$. To see why in (**) if $f(x_k^*)$ is positive, then the product $f(x_k^*)\Delta x$ represents the area of the rectangle with height $f(x_k^*)$ and base Δx. However, if $f(x_k^*)$ is negative, then the product $f(x_k^*)\Delta x$ is the negative of the area of the rectangle with height $|f(x_k^*)|$ and base Δx.

For example, in Figure below, we have

$$\sum_{k=1}^{6} f(x_k^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots\cdots + f(x_6^*)\Delta x$$

$$= A_1 + A_2 - A_3 - A_4 + A_5 + A_6 = (A_1 + A_2 + A_5 + A_6) - (A_3 + A_4)$$

$$= (\text{area above } [a,b]) - (\text{area below } [a,b])$$



The definite integral (التكامل المحدد)

Let $x_1, x_2, \ldots\ldots, x_{n-1}$ be any numbers in $[a,b]$ such that

$$a < x_1 < x_2 < \cdots\cdots < x_{n-1} < b$$

These values divide $[a,b]$ into n subintervals

$[a, x_1], [x_1, x_2], \text{------}, [x_{n-1}, b]$

whose different lengths, as usual, we denote by

$\Delta x_1, \Delta x_2, \text{------}, \Delta x_n$

And let us denote the largest of these by the symbol

$\max \Delta x_k$ (read " the maximum of the Δx_k's ")

As $\max \Delta x_k \longrightarrow 0$, we get

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

From above we get the following definition

Definition: If a function f is defined on a closed interval $[a, b]$, then the definite integral of f from a to b, denoted by $\int_a^b f(x)\,dx$, is defined to be

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k \quad \left(\begin{array}{l}\text{Which is called} \\ \text{Riemann sum}\end{array}\right)$$

provided the limit exists

The Fundamental Theorem of Integration (المبرهنة الاساسية للتكامل)

Theorem (The Fundamental Theorem of integration): If $f(x)$ is continuous on $[a, b]$ and $F(x)$ is any indefinite integral of f on $[a, b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

Example: Evaluate $\int_1^2 x\,dx$

Solution: The function $F(x) = \frac{x^2}{2}$ is an indefinite integral of $f(x) = x$

$$\therefore \int_1^2 x\,dx = F(2) - F(1) = \frac{(2)^2}{2} - \frac{(1)^2}{2} = 2 - \frac{1}{2} = \frac{3}{2}$$

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properties of the definite integral (خصائص التكامل المحدد)

Theorem: If f and g are continuous on $[a,b]$ and if c is a constant, then

① $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

② $\int_a^b [f(x) \mp g(x)]\,dx = \int_a^b f(x)\,dx \mp \int_a^b g(x)\,dx$

③ $\int_a^a f(x)\,dx = 0$ ④ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

⑤ $\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^b f(x)\,dx$, where $a \le c_1 \le b$

Examples:

① $\int_1^1 x^2\,dx = \left.\frac{x^3}{3}\right]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$

② $\int_4^0 x\,dx = \left.\frac{x^2}{2}\right]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$ and $-\int_0^4 x\,dx = -\left.\frac{x^2}{2}\right]_0^4 = -8$

$$\therefore \int_4^0 x\,dx = -\int_0^4 x\,dx = -8$$

③ ⓐ ~~Evaluate $\int_0^6 f(x)dx$ if $f(x)=\begin{cases} x^2 & x\le 2 \\ 3x-2 & x\ge 2\end{cases}$~~

Solution:

$$\int_0^6 f(x)dx=\int_0^2 f(x)dx+\int_2^6 f(x)dx=\int_0^2 x^2dx+\int_2^6 (3x-2)\,dx$$

$$=\left[\frac{x^3}{3}\right]_0^2+\left[\frac{3x^2}{2}-2x\right]_2^6=\left(\frac{8}{3}-0\right)+(42-2)=\frac{8}{3}+40=\frac{128}{3}$$

ⓑ Evaluate $\int_{-1}^{2} |x|\,dx$ (ch)

Evaluating definite integral by substitution

Examples: Evaluate ① $\int_0^2 2x(x^2+1)^3\,dx$ ② $\int_0^{\frac{\pi}{8}} \sin^5 2x\cos 2x\,dx$

Solution:

① الطريقة الأولى

$$\int_0^2 2x(x^2+1)^3\,dx=\left.\frac{(x^2+1)^4}{4}\right]_0^2=\frac{5^4}{4}-\frac{1}{2}=\frac{625}{4}-\frac{1}{4}=\frac{624}{4}=156$$

$\boxed{\text{or}}$

Let $u=x^2+1 \Rightarrow du=2x\,dx$

$\therefore$ If $x=0 \Rightarrow u=(0)^2+1=1 \Rightarrow u=1$

If $x=2 \Rightarrow u=(2)^2+1=5 \Rightarrow u=5$

$$\therefore \int_0^2 2x(x^2+1)^3\,dx=\int_1^5 u^3\,du=\left.\frac{u^4}{4}\right]_1^5=\frac{5^4}{4}-\frac{1}{4}=\frac{625-1}{4}=156$$

② $$\int_0^{\frac{\pi}{8}} \sin^5 2x\cos 2x\,dx=\frac{1}{2}\int_0^{\frac{\pi}{8}} 2\sin^5 2x\cos 2x\,dx$$

$$= \frac{1}{2}\,\frac{\sin^6(2x)}{6}\Big]_0^{\frac{\pi}{8}} = \frac{1}{12}\left[\sin^6\left(\frac{\pi}{4}\right) - \sin^6(0)\right] = \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^6 - (0)^6\right]$$

$$= \frac{1}{12}\left[\frac{1}{8} - 0\right] = \frac{1}{12}\left[\frac{1}{8}\right] = \frac{1}{96}$$

$\boxed{\text{or}}$

Let $u = \sin 2x \rightarrow du = 2\cos 2x\, dx \rightarrow \frac{du}{2} = \cos 2x\, dx$

$\therefore$ If $x = 0 \rightarrow u = \sin(0) = 0 \rightarrow u = 0$

If $x = \frac{\pi}{8} \rightarrow u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \rightarrow u = \frac{1}{\sqrt{2}}$

$$\therefore \int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x\, dx = \int_0^{\frac{1}{\sqrt{2}}} u^5 \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} u^5\, du$$

$$= \frac{1}{2}\,\frac{u^6}{6}\Big]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{12}\left[\left(\frac{1}{\sqrt{2}}\right)^6 - (0)^6\right] = \frac{1}{12}\left[\frac{1}{8}\right] = \frac{1}{96}$$

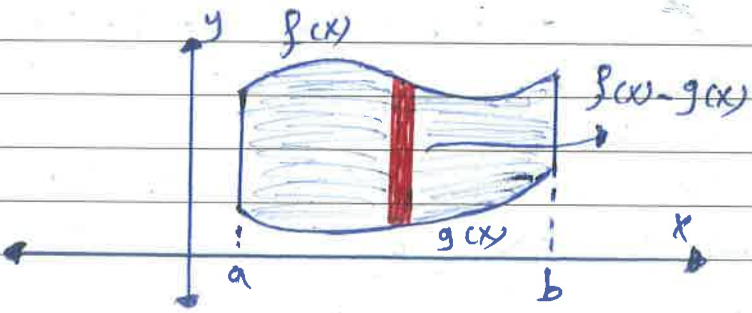
Applications of the definite integral (تطبيقات التكامل المحدد)

Area between two curves (المساحة بين منحنيين)

① Area between $y = f(x)$ and $y = g(x)$

If f and g are continuous functions on the interval $[a,b]$, and if $f(x) \geq g(x)$ for all x in $[a,b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

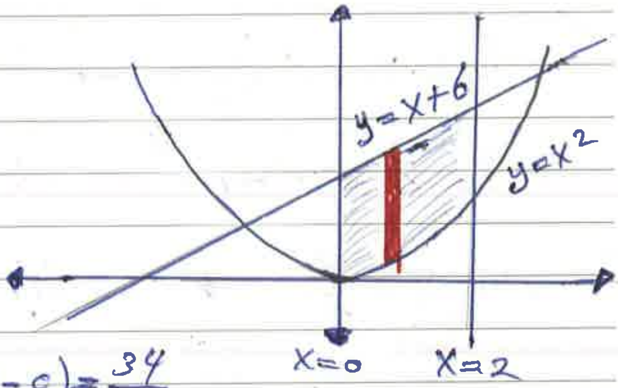
$$A = \int_a^b [f(x) - g(x)]\, dx$$



Examples:

① Find the area of the region bounded above by $y=x+6$, bounded below by $y=x^2$, and bounded on the sides by the lines $x=0$ and $x=2$

Solution:



$f(x)=x+6$ and $g(x)=x^2$

$x=a=0$ and $x=b=2$

$$\therefore A=\int_0^2 [(x+6)-x^2]\,dx$$

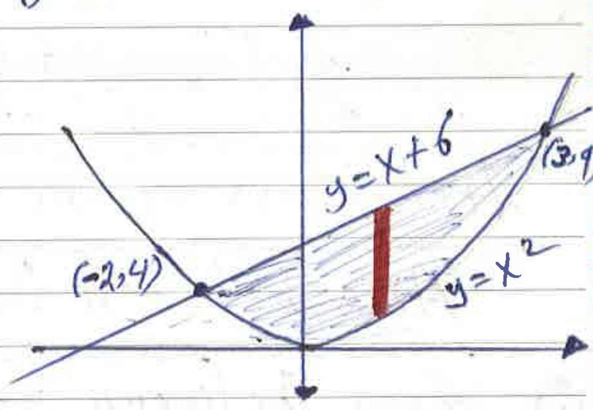
$$=\frac{x^2}{2}+6x-\frac{x^3}{3}\Big]_0^2=\left(2+12-\frac{8}{3}\right)-(0+0-0)=\frac{34}{3}$$

② Find the area of the region enclosed by $y=x^2$ and $y=x+6$

Solution:

نجد أولاً نقاط تقاطع المستقيم $y=x+6$

مع المنحنى $y=x^2$ كالآتي:



$y=x+6$ and $y=x^2$

$\Rightarrow x^2=x+6 \Rightarrow x^2-x-6=0$

$\Rightarrow (x+2)(x-3)=0 \Rightarrow x=-2\ (y=4)$

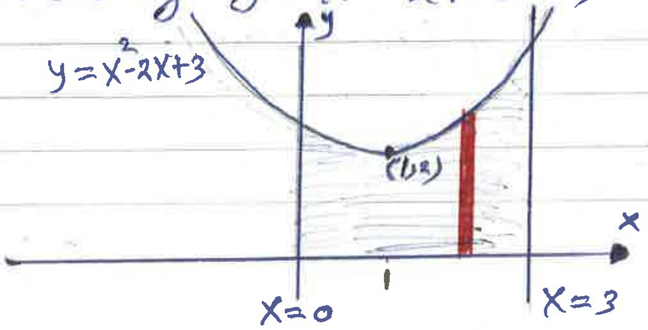
and $x=3\ (y=9)$

$$\therefore A=\int_{-2}^{3}[(x+6)-x^2]\,dx=\frac{x^2}{2}+6x-\frac{x^3}{3}\Big]_{-2}^{3}$$

$$=\left(\frac{9}{2}+18-9\right)-\left(2-12+\frac{8}{3}\right)=\left(\frac{9}{2}+9+10-\frac{8}{3}\right)=\frac{125}{6}$$

③ Find the area of the region enclosed by $y=x^2-2x+3$, x-axis, $x=0$ and $x=3$

Solutions



$y=x^2-2x+3=(x^2-2x+1)+3-1$

$\Rightarrow y=(x-1)^2+2$ (قطع مكافئ)

$$A=\int_0^3[(x^2-2x+3)-(0)]\,dx=\int_0^3(x^2-2x+3)\,dx=\frac{x^3}{3}-x^2+3x\Big]_0^3$$

$$=(9-9+9)-(0-0+0)=9$$

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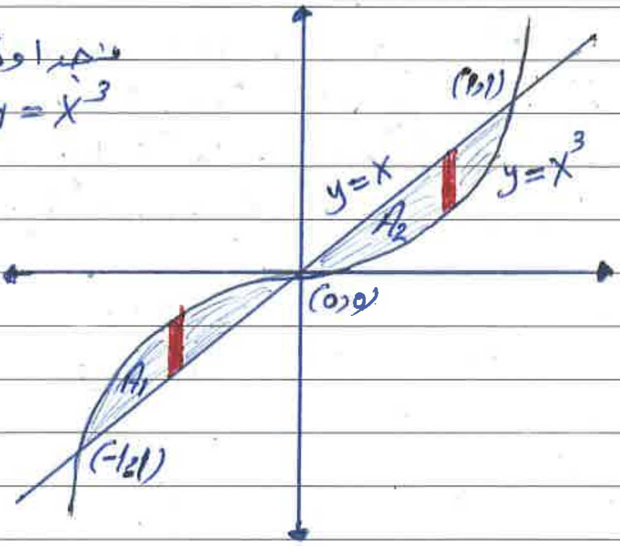
④ Find the area of the region enclosed by $y=x^3$ and $y=x$

Solution

نجد أولاً نقاط تقاطع المستقيم $y=x$ مع المنحنى $y=x^3$ كما يلي:



$\therefore x^3=x \rightarrow x^3-x=0 \rightarrow x(x^2-1)=0$

$\rightarrow x(x+1)(x-1)=0 \rightarrow x=0\ (y=0)$

and $x=-1\ (y=-1)$ and $x=1\ (y=1)$

$$\therefore A=A_1+A_2=\int_{-1}^{0}(x^3-x)\,dx+\int_0^1(x-x^3)\,dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$=\left[(0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]+\left[\left(\frac{1}{2}-\frac{1}{4}\right)-(0)\right]=-\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4}=\frac{1}{2}$$

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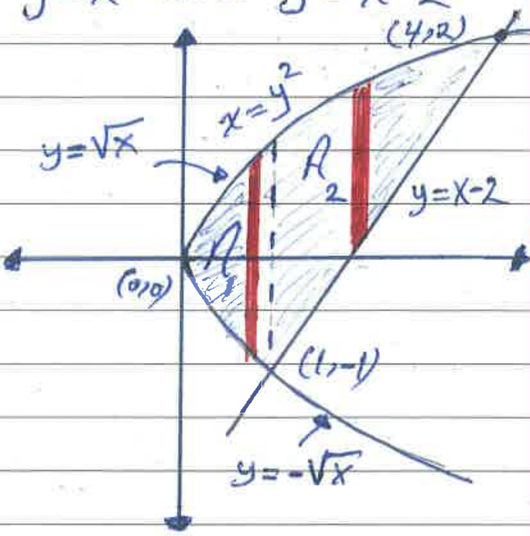
⑤ Find the area of the region enclosed by $y^2=x$ and $y=x-2$

Solution

نجد أولاً نقاط تقاطع المستقيم $y=x-2$ مع المنحنى $x=y^2$ كما يلي:



$x=y^2$ and $x=y+2 \rightarrow y^2=y+2$

$\rightarrow y^2-y-2=0 \rightarrow (y+1)(y-2)=0 \rightarrow$

$y=-1\ (x=1)$ and $y=2\ (x=4)$

$$A_1=\int_0^1[\sqrt{x}-(-\sqrt{x})]\,dx=2\int_0^1\sqrt{x}\,dx=2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$=2\left[\frac{2}{3}-0\right]=\frac{4}{3}$$

$$A_2=\int_1^4[\sqrt{x}-(x-2)]\,dx=\int_1^4(\sqrt{x}-x+2)\,dx=\frac{2}{3}x^{\frac{3}{2}}-\frac{x^2}{2}+2x\Big]_1^4$$

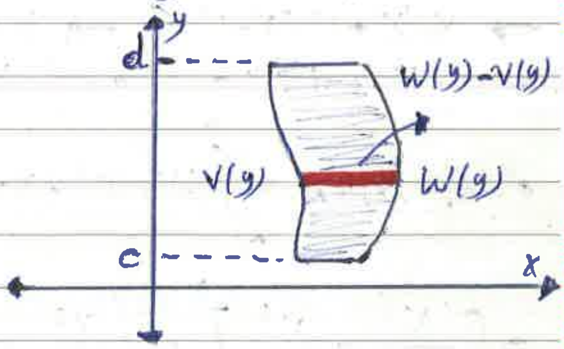
$$= \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) = \frac{14}{3} - \frac{3}{2} = \frac{28-9}{6} = \frac{19}{6}$$

$$\therefore A = A_1 + A_2 = \frac{4}{3} + \frac{19}{6} = \frac{8+19}{6} = \frac{27}{6} = \frac{9}{2}$$

② Area between $x = v(y)$ and $x = w(y)$

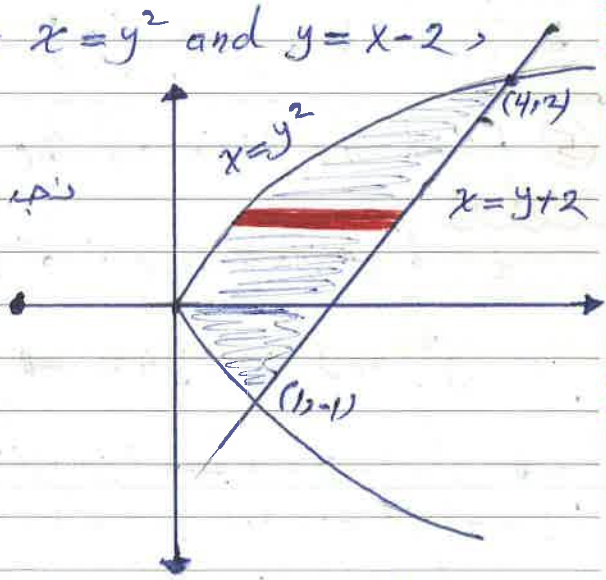
If w and v are continuous functions on the interval $[c,d]$, and if $w(y) \geq v(y)$ for all y in $[c,d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is:-

$$A = \int_c^d [w(y) - v(y)]\,dy$$



Examples

① Find the area of the region enclosed by $x = y^2$ and $y = x - 2$, integrating with respect to y.



Solution

نجد اولاً نقاط تقاطع المنحنيين $y = x - 2$ و $x = y^2$

$y^2 = x$ and $x = y + 2$

$y^2 = y + 2 \to y^2 - y - 2 = 0 \to (y+1)(y-2) = 0$

$\to y = -1\ (x = 1)$ and $y = 2\ (x = 4)$

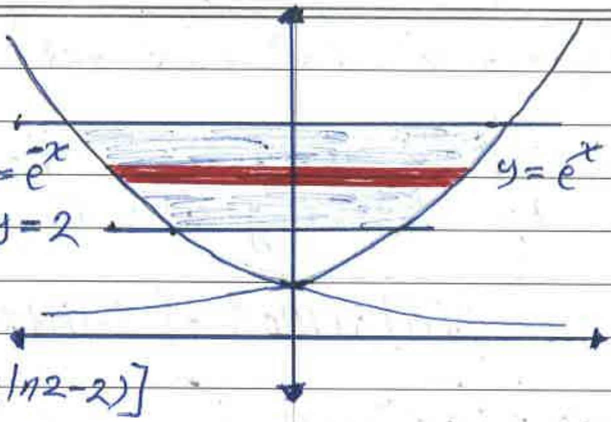
$$A = \int_{-1}^{2} [(y+2) - y^2]\,dy = \int_{-1}^{2} (y + 2 - y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big]_{-1}^{2} = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = 8 - 3 - \frac{1}{2} = \frac{9}{2}$$

نفس الجواب السابق (مثال ⑤ اعلاه)

② Find the area of the region enclosed by $y = e^{x}$, $y = e^{-x}$, $y = 2$ and $y = 4$

Solution



$$A=\int_2^4[\ln y-(-\ln y)]\,dy=2\int_2^4 \underset{u}{\ln y}\,\underset{dv}{dy}$$

$$=2\left[y\ln y-\int y\cdot\frac{1}{y}\,dy\right]=2\left[y\ln y-y\right]_2^4$$

$$=2\left[(4\ln 4-4)-(2\ln 2-2)\right]=2\left[(4\ln 2^2-4)-(2\ln 2-2)\right]$$

$$=2\left[8\ln 2-4-2\ln 2+2\right]=2\left[6\ln 2-2\right]=12\ln 2-4$$

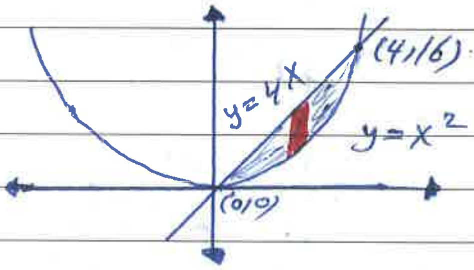
③ Find the area of the region enclosed by $y=x^2$ and $y=4x$ by integrating ⓐ with respect to x ⓑ with respect to y

Solution

نجد اولاً نقاط تقاطع المستقيم $y=4x$ مع المنحنى $y=x^2$

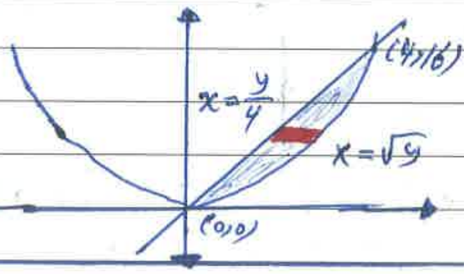
$x^2=4x \Rightarrow x^2-4x=0 \Rightarrow x(x-4)=0 \Rightarrow x=0\ (y=0)$ and $x=4\ (y=16)$

ⓐ with respect to x



$$A=\int_0^4(4x-x^2)\,dx=2x^2-\frac{x^3}{3}\Big]_0^4=32-\frac{64}{3}=\frac{32}{3}$$

ⓑ with respect to y



$$A=\int_0^{16}\left(\sqrt{y}-\frac{y}{4}\right)dy=\frac{2}{3}y^{\frac{3}{2}}-\frac{y^2}{8}\Big]_0^{16}$$

$$=\frac{128}{3}-32=\frac{128-96}{3}=\frac{32}{3}$$ (نفس الجواب السابق)

Exercises

Q1) Find the area of the region enclosed by :-

① $y=x^3$, x-axis, $x=1$ and $x=3$

② $y=4-x^2$ and x-axis

③ $y=x^2-5$, x-axis, $x=-1$ and $x=2$

~~(4) $y=x^2$, $y=\sqrt{x}$, $x=\frac{1}{4}$ and $x=1$~~

(5) $y=x^3-6x^2+8x$ and x-axis

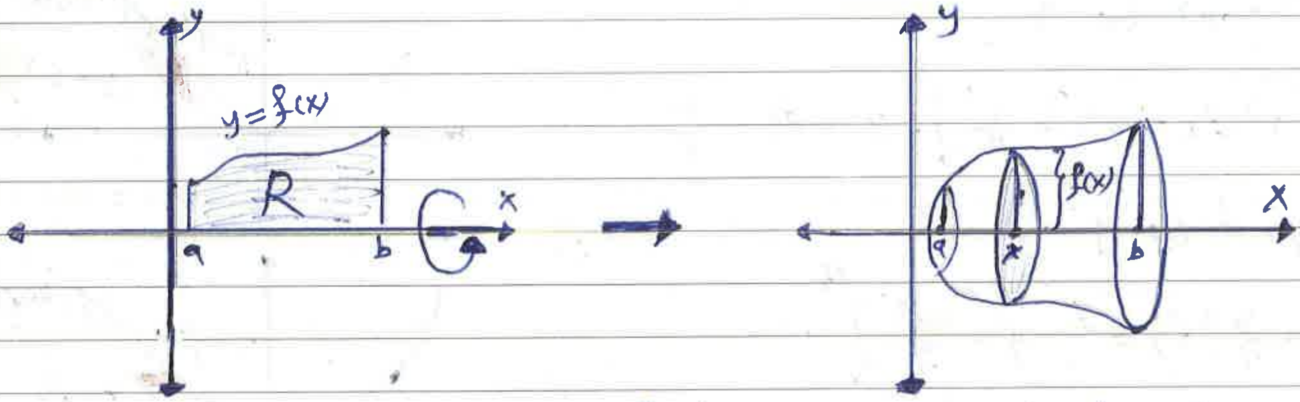
(Q2) Find the area of the region enclosed by $y^2=4x$ and $y=2x-4$ by integrating (a) with respect to x (b) with respect to y

Volumes of Solids of Revolution (حجم الجسم المتولد من الدوران)

(1) Disc Method

Suppose that f is nonnegative and continuous on $[a,b]$. If we revolve the region below the graph of f about the x-axis, we obtain a solid. The volume of this solid is given by the formula

$$V=\int_a^b \pi\,[f(x)]^2\,dx$$

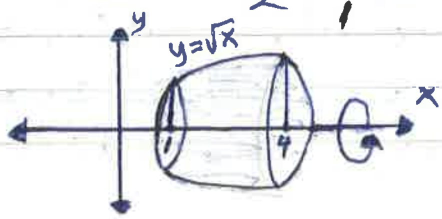


Example: Find the volume of the solid obtained when the region under the curve $y=\sqrt{x}$ over the interval $[1,4]$ is revolved about the x-axis

Solution:

$$V=\int_a^b \pi\,[f(x)]^2\,dx=\int_1^4 \pi\,[\sqrt{x}]^2\,dx=\int_1^4 \pi x\,dx=\frac{\pi x^2}{2}\Big|_1^4=$$

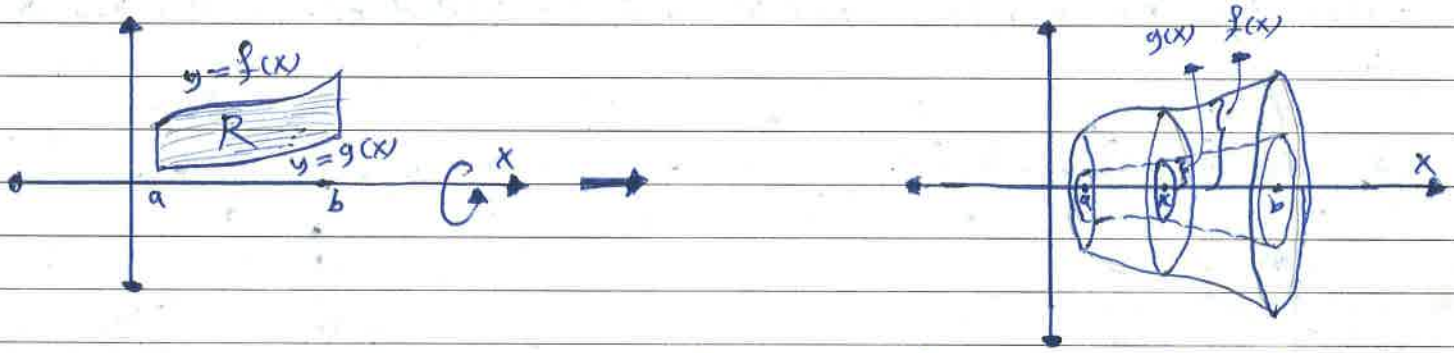
$$=8\pi-\frac{\pi}{2}=\frac{15\pi}{2}$$



② Washer method

suppose that f and g are nonnegative continuous functions with $f(x) \geq g(x)$ for all x in $[a,b]$. If we revolve the region R about the x-axis, we obtain a solid. The volume of this solid is given by the formula

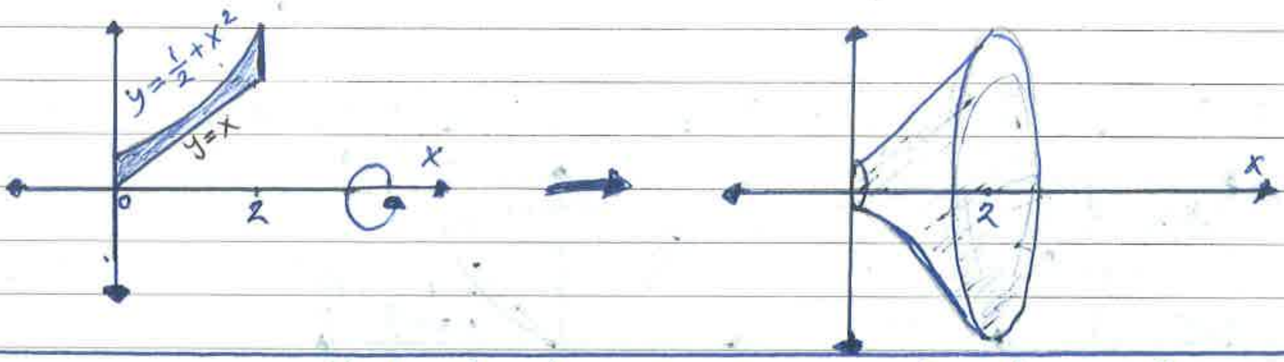
$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx$$



Example: Find the volume of the solid generated when the region enclosed by $y = \frac{1}{2} + x^2$, $y = x$, $x = 0$ and $x = 2$ is revolved about the x-axis.

$$A = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right) dx = \int_0^2 \pi\left(\left[\tfrac{1}{2} + x^2\right]^2 - x^2\right) dx$$

$$= \int_0^2 \pi\left(\frac{1}{4} + x^4\right) dx = \pi\left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2 = \pi\left[\frac{1}{2} + \frac{32}{5}\right] = \frac{69}{10}\pi$$



suppose now that the boundaries are functions of y rather than x.

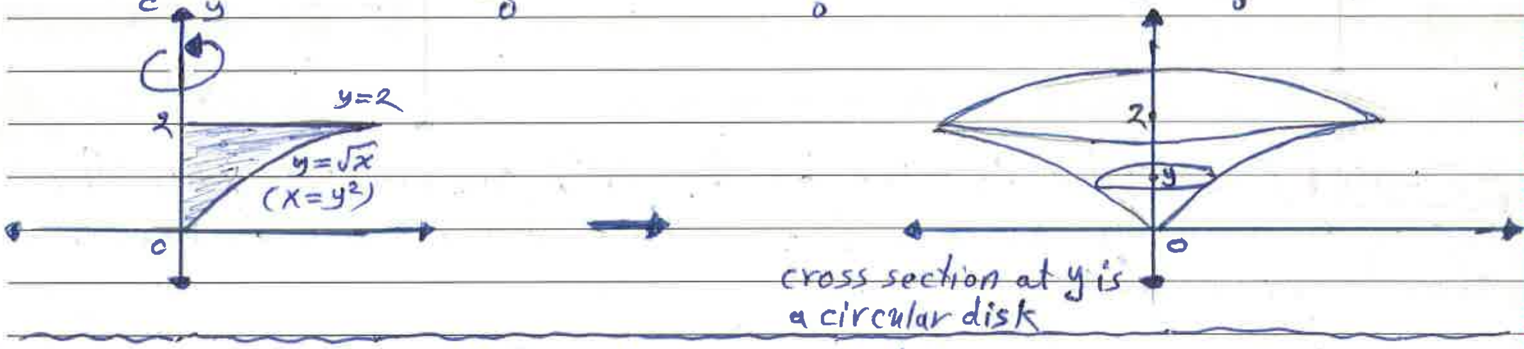
~~By revolving R about the y-axis, we obtain a solid. The volume~~ of this solid is given by:

$V=\int_c^d \pi\,[u(y)]^2\,dy$	$V=\int_c^d \pi\left([u(y)]^2-[v(y)]^2\right)dy$
Disc	washer

Examples: ① Find the volume of the solid generated when the region enclosed by $y=\sqrt{x}$, $y=2$, and $x=0$ is revolved about the y-axis

Solution

$$V=\int_c^d \pi\,[u(y)]^2\,dy=\int_0^2 \pi\,[y^2]^2\,dy=\int_0^2 \pi\,y^4\,dy=\frac{\pi y^5}{5}\Big]_0^2=\frac{32\pi}{5}$$



② Find the volume of the solid generated by revolving the region between $y=x^2$ and $y=2x$ ⓐ about the x-axis ⓑ about the y-axis

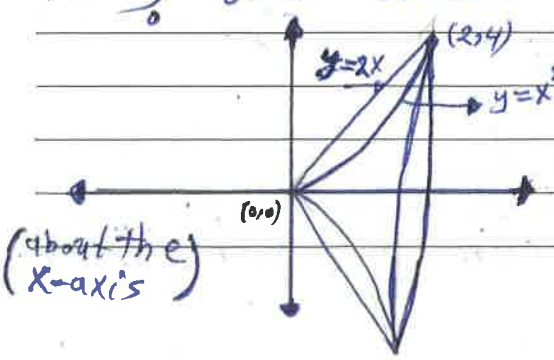
Solution

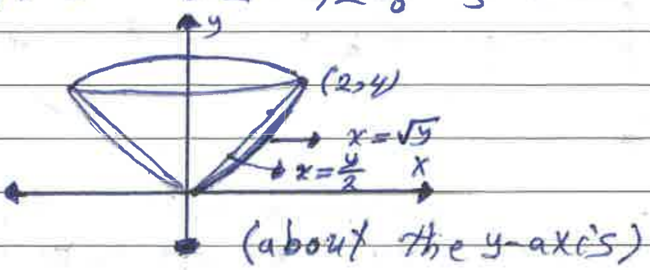
ⓐ

$$V=\int_0^2 \pi\left[(2x)^2-(x^2)^2\right]dx=\pi\int_0^2 (4x^2-x^4)\,dx=\pi\left[\frac{4}{3}x^3-\frac{x^5}{5}\right]_0^2=\frac{64}{15}\pi$$

ⓑ

$$V=\int_0^4 \pi\left[(\sqrt{y})^2-\left(\frac{y}{2}\right)^2\right]dy=\pi\int_0^4 \left(y-\frac{y^2}{4}\right)dy=\pi\left[\frac{y^2}{2}-\frac{y^3}{12}\right]_0^4=\frac{8}{3}\pi$$





Improper Integrals

Definitions: ① The improper integral of f over the interval $[a,+\infty)$ is defined as:-

$$\int_a^{+\infty} f(x)\,dx = \lim_{l\to+\infty}\int_a^{l} f(x)\,dx$$

② The improper integral of f over the interval $(-\infty,b]$ is defined as:-

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{k\to-\infty}\int_k^{b} f(x)\,dx$$

In ① and ② the integral is said to converge if the limit exists and diverge if it does not.

③ The improper integral of f over the interval $(-\infty,+\infty)$ is defined as:-

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

The improper integral is said to converge if both terms converge and diverge if either term diverges.

Examples

① $\displaystyle\int_1^{+\infty}\frac{dx}{x^3} = \lim_{l\to+\infty}\int_1^{l}\frac{dx}{x^3} = \lim_{l\to+\infty}\left[\frac{-1}{2x^2}\right]_1^{l} = \lim_{l\to+\infty}\left[\frac{-1}{2l^2}+\frac{1}{2}\right] = \frac{1}{2}$

$\boxed{\text{or}}$ $\displaystyle\int_1^{+\infty}\frac{dx}{x^3} = \left[\frac{-1}{2x^2}\right]_1^{+\infty} = 0+\frac{1}{2} = \frac{1}{2}$ (converge)

② $\int_1^{+\infty} \frac{dx}{x} = \lim_{l \to +\infty} \int_1^{l} \frac{dx}{x} = \lim_{l \to +\infty} \left[\ln|x|\right]_1^{l} = \lim_{l \to +\infty} \left[\ln|l| - \ln|1|\right]$

$= \lim_{l \to +\infty} \ln|l| = \ln|\infty| = \infty$ (diverge)

$\boxed{\text{or}}$ $\int_1^{+\infty} \frac{dx}{x} = \left[\ln|x|\right]_1^{+\infty} = \ln|\infty| - \ln|1| = \infty - 0 = \infty$ (diverge)

③ $\int_{-\infty}^{0} e^{x}\, dx = \lim_{k \to -\infty} \int_k^{0} e^{x}\, dx = \lim_{k \to -\infty} \left[e^{x}\right]_k^{0} = \lim_{k \to -\infty} \left[1 - e^{k}\right]$

$= 1 - e^{-\infty} = 1 - 0 = 1$ (converge)

$\boxed{\text{or}}$ $\int_{-\infty}^{0} e^{x}\, dx = \left[e^{x}\right]_{-\infty}^{0} = e^{0} - e^{-\infty} = 1 - 0 = 1$ (converge)

④ $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$\therefore \int_0^{\infty} \frac{dx}{1+x^2} = \lim_{l \to \infty} \int_0^{l} \frac{dx}{1+x^2} = \lim_{l \to \infty} \left[\tan^{-1} x\right]_0^{l}$

$= \lim_{l \to \infty} \left[\tan^{-1} l - \tan^{-1} 0\right] = \lim_{l \to \infty} \left[\tan^{-1}(l) - 0\right] = \lim_{l \to \infty} \tan^{-1} l = \frac{\pi}{2}$

Also, $\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{k \to -\infty} \int_k^{0} \frac{dx}{1+x^2} = \lim_{k \to -\infty} \left[\tan^{-1} x\right]_k^{0}$

$= \lim_{k \to -\infty} (-\tan^{-1} k) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

$$② \int_{1}^{+\infty} \frac{dx}{x} = \lim_{l \to +\infty} \int_{1}^{l} \frac{dx}{x} = \lim_{l \to +\infty} \left[\ln|x|\right]_{1}^{l} = \lim_{l \to +\infty} \left[\ln|l| - \ln|1|\right]$$

$$= \lim_{l \to +\infty} \ln|l| = \ln|\infty| = \infty \quad (\text{diverge})$$

$$\boxed{\text{or}} \quad \int_{1}^{+\infty} \frac{dx}{x} = \left[\ln|x|\right]_{1}^{+\infty} = \ln|\infty| - \ln|1| = \infty - 0 = \infty \quad (\text{diverge})$$

$$③ \int_{-\infty}^{0} e^{x}\,dx = \lim_{k \to -\infty} \int_{k}^{0} e^{x}\,dx = \lim_{k \to -\infty} \left[e^{x}\right]_{k}^{0} = \lim_{k \to -\infty} \left[1 - e^{k}\right]$$

$$= 1 - e^{-\infty} = 1 - 0 = 1 \quad (\text{converge})$$

$$\boxed{\text{or}} \quad \int_{-\infty}^{0} e^{x}\,dx = \left[e^{x}\right]_{-\infty}^{0} = e^{0} - e^{-\infty} = 1 - 0 = 1 \quad (\text{converge})$$

$$④ \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{1+x^2} = \lim_{l \to \infty} \int_{0}^{l} \frac{dx}{1+x^2} = \lim_{l \to \infty} \left[\tan^{-1} x\right]_{0}^{l}$$

$$= \lim_{l \to \infty} \left[\tan^{-1} l - \tan^{-1} 0\right] = \lim_{l \to \infty} \left[\tan^{-1}(l) - 0\right] = \lim_{l \to \infty} \tan^{-1} l = \frac{\pi}{2}$$

$$\text{Also,} \quad \int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{k \to -\infty} \int_{k}^{0} \frac{dx}{1+x^2} = \lim_{k \to -\infty} \left[\tan^{-1} x\right]_{k}^{0}$$

$$= \lim_{k \to -\infty} \left(-\tan^{-1} k\right) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$