linear Algebra 1

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Lecture 1

Filed with vectors

جبر خطي
مرحلة ٢/ علوم رياضيات

Linear Algebra with Applications I

References

1) Introductary Linear Algebra with Applications, by Bernard Kolman.
2) Elementary Linear Algebra, by Howard Anton.
3) Linear Algebra, by Fraleigh
4) Theory and Problems of Linear Algebra, by Seymour Lipschutz.
5) 3000 Solved Problems in Linear Algebra, by Seymour Lipschutz.
6) Introduction to Linear Algebra, by Franz Hohn.

§ 1 : Fields

Definition : A binary operation $*$ on a set X is a function $* : X \times X \to X$.

Definition : A set F with two binary operations $+$ and $\cdot$ defined on F is said to be a field if the following properties are satisfied :

1) $a+b \in F \quad \forall a,b \in F$,
2) $a+b = b+a \quad \forall a,b \in F$,
3) $a+(b+c) = (a+b)+c \quad \forall a,b,c \in F$,
4) There exists an element $z \in F$ such that $a+z = z+a \quad \forall a \in F$,
5) $\forall a \in F$ there is an element denoted by $-a \in F$ such that $a+(-a) = (-a)+a = z$,
6) $a \cdot b \in F \quad \forall a,b \in F$,
7) $a \cdot b = b \cdot a \quad \forall a,b \in F$,
8) $a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad \forall a,b,c \in F$,
9) There exists an element $u \in F$ such that $u \neq z$ and $a \cdot u = u \cdot a = a \quad \forall a \in F$,
10) $\forall a \in F - \{z\}$ there exists an element denoted by $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = u$,
11) $a \cdot (b+c) = (a \cdot b) + (a \cdot c) \quad \forall a,b,c \in F$.

3. Examples:

1) The set R of all real numbers with the usual addition $+$ and the usual multiplication $\cdot$ of real numbers is a field.

2) The set $Q = \{ \frac{p}{q} \mid p$ and q are integers and $q \neq 0 \}$ of all rational numbers with the usual addition $+$ and the usual multiplication $\cdot$ of rational numbers is a field.

3) The set $C = \{ a+bi \mid a$ and b are real numbers$\}$ of all complex numbers with the usual addition $+$ and the usual multiplication $\cdot$ of the complex numbers is a field (notice that $i \cdot i = -1$ and $(a+bi)^{-1} = (a-bi)/(a^2+b^2)$).

4) The set R of all real numbers with the two binary opertions $*$ and $\circ$ defined by $a*b = a+b+1$ and $a \circ b = a \cdot b + a + b$ $\forall a, b \in F$ is a field.

5) The set $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$ with the two binary operations $+_p$ (the addition of integers modulo p) and $\cdot_p$ (the multiplication of integers modulo p) is a field where p is any prime number.

6) The set $F_3 = \{4, 8, 12\}$ and $+'_{12}$ and $\cdot'_{12}$ be the two binary operations defined on F_3 as

4. fallows :

$+_{12}$	4	8	12
4	8	12	4
8	12	4	8
12	4	8	12

$\cdot_{12}$	4	8	12
4	4	8	12
8	8	4	12
12	12	12	12

is a field

§2 : Vectors in the Plane

Definition: A vector in the plane is a 2×1 matrix $X = \binom{x}{y}$ where x and y are real numbers. x is called the first component of X and y is called the second compotent .

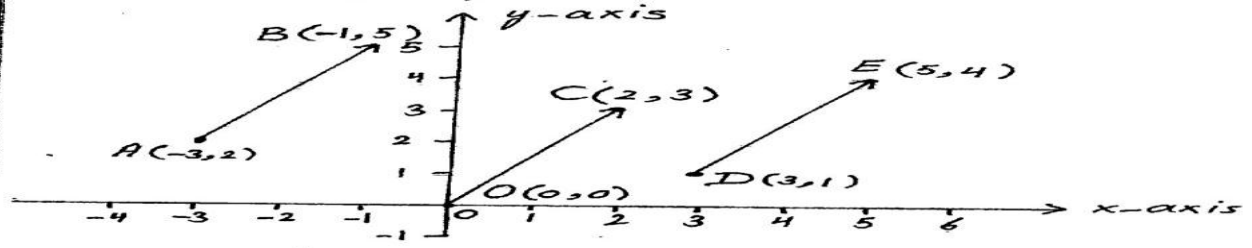
Remarks :

1) We can also write the vector X as $X = (x, y)$.
2) The set of all vectors in the plane (the Euclidean plane) is denoted by R^2 .
3) The vector $X = \binom{x}{y}$ (or $X = (x, y)$) is represented by the line segment with initial point $O(0,0)$ and the terminal point $P(x,y)$. The initial point $O(0,0)$ is called the tail of the vector X and the terminal point $P(x,y)$ is called the head

5 of the vector X, and X is denoted by $\overrightarrow{OP}$
The vector $X = \binom{x}{y}$ is also represented by any line segment with initial point $A = (a_1, a_2)$ and terminal point $B\,(a_1 + x, a_2 + y)$ and A is the tail of the vector X and B is the head of the vector X, and X is then will be denoted by $\overrightarrow{AB}$.

4) Any vector $X = \binom{x}{y}$ (or $X = (x, y)$) has length or magnitude denoted by $\|X\|$ which is equal $\sqrt{x^2+y^2}$ (i.e. $\|X\| = \sqrt{x^2+y^2}$).

Example: The vector $X = \binom{2}{3}$ can be represented by any of the following three vectors shown in the following figure



Example: Find the length of the vector $X = (7, -3)$.

Solution: The length of the vector X is

6. $\|X\| = \sqrt{7^2+(-3)^2} = \sqrt{49+9} = \sqrt{58}$.

Example: Find the length of the vector $\overrightarrow{PQ}$ where P is the point $(4,3)$ and Q is the point $(2,-9)$.

Solution: $\overrightarrow{PQ} = (2-4, -9-3) = (-2, -12)$.

$\therefore \|\overrightarrow{PQ}\| = \sqrt{(-2)^2+(-12)^2} = \sqrt{4+144} = \sqrt{148}$.

Vector Operations in R^2

Definition: Let $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ be any two vectors in the Euclidean plane. The sum of the vectors X and Y is the vector (x_1+x_2, y_1+y_2) and is denoted by $X+Y$.

Example: Find $X+Y$ where X and Y are the vectors $X=(7,8)$ and $Y=(3,-2)$.

Solution: $X+Y = (7+3, 8-2) = (10, 6)$.

Definition: If $X=(x,y)$ be any vector in R^2 and c is any scalar ($c \in R$), then the scalar multiple cX of X by c is the vector (cx, cy) (i.e. $cX = (cx, cy)$).

Example: Find $7X$ where $X = (2,3)$.

7. <u>Solution</u> : $7X = 7(2,3) = (14, 21)$.

<u>Remarks</u> :

1) The vector $O(0,0)$ is called the zero vector which satisfies that $X + O = X \quad \forall X \in R^2$.

2) $\forall X = (x, y) \in R^2$, we have $(-1)X = -1(x,y) = (-x, -y)$. We write $(-1)X$ as $-X$.

3) $\forall X, Y \in R^2$ we write $X - Y$ instead of $X + (-Y)$.

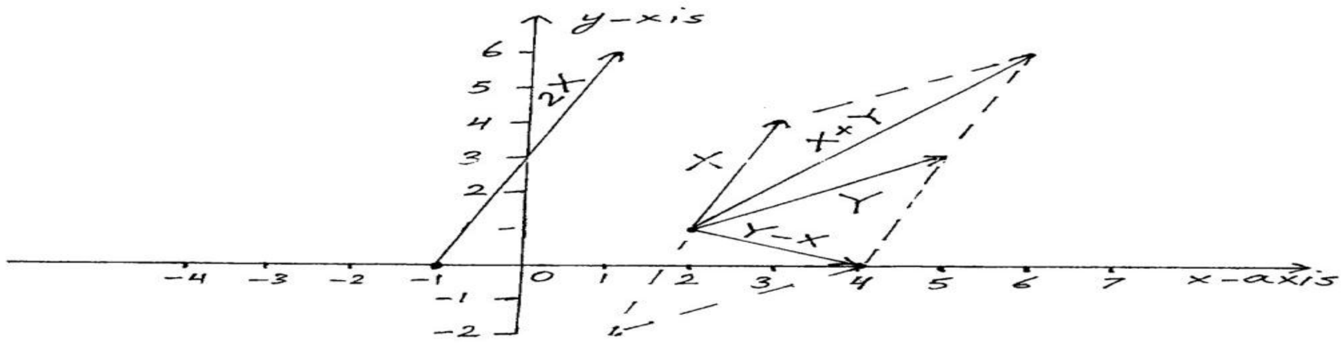
4) $\forall X \in R^2$, we have $X - X = O$.

<u>Example</u> : Let $X = (1,3)$ and $Y = (3,2)$ and $c = 2$. Find cX, $X + Y$, $Y - X$ and show the answer in a figure.

<u>Solution</u> : $cX = 2X = 2(1,3) = (2,6)$.

$X + Y = (1,3) + (3,2) = (4,5)$.

$Y - X = (3,2) - (1,3) = (2,-1)$.



$\therefore\ \theta = 41.82^\circ$ is the angle between X and Y.

Definition: A unit vector is a vector whose length is 1. If X is any nonzero vector, then the vector $U_X = \frac{X}{\|X\|}$ is the unit vector in the direction of X.

Example: Let $X = (8, -6)$. Find the unit vector in the direction of X.

Solution: $\|X\| = \sqrt{8^2 + (-6)^2} = \sqrt{64+36} = \sqrt{100} = 10.$

Thus $U_X = \frac{1}{10}(8, -6) = (0.8, -0.6)$.

Remarks:

1) The unit vector along the positive x-axis is $(1, 0)$ and is denoted by i (i.e. $i = (1, 0)$).
2) The unit vector along the positive y-axis is $(0, 1)$ and is denoted by j (i.e. $j = (0, 1)$).
3) The two nonzero vectors X and Y are perpendicular (or orthogonal) iff $X \cdot Y = 0$.
4) i and j are orthogonal vectors since $i \cdot j = (1, 0) \cdot (0, 1) = 0$
5) We can write any vector $X = (x, y)$ as $X = xi + yj$ since $X = (x, y) = (x, 0) + (0, y)$

$= x(1,0) + y(0,1) = xi + yj$.

6) i and j are called the standard unit vectors in R^2.

Example: Write the vector $X = (7,12)$ in terms of i and j .

Solution: $X = (7,12) = 7i + 12j$.

Remarks:

1) If the set F with the two binary operations $+$ and $\cdot$ is a field, then this field will be denoted by $(F, +, \cdot)$

2) If $(F, +, \cdot)$ is a field then the set of all vectors $X = (x, y)$ where $x, y \in F$ is denoted by F^2 .

Example: For the field $(\mathbb{Z}_7, +_7, \cdot_7)$, let $X = (3,2)$ and $Y = (5,6)$ be two vectors in $\mathbb{Z}_7^2$. Find $3X + 4Y$.

Solution:

$$
\begin{aligned}
3X + 4Y &= 3(3,2) + 4(5,6) \\
&= (3 \cdot_7 3, 3 \cdot_7 2) + (4 \cdot_7 5, 4 \cdot_7 6) \\
&= (2, 6) + (6, 3) \\
&= (2 +_7 6, 6 +_7 3) \\
&= (1, 2) .
\end{aligned}
$$

Caution: The definition of the inner product (or the dot product) given in page 8 is not applicable for F^2 if F is neither R nor Q .