linear Algebra 1

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Lecture 2

Vector space

§3: Vector Spaces

Definition: Let $(F,+,\cdot)$ be a field.
A vector space V over F is a set V together with a binary operation $\oplus$ and an operation of scalar multiplication of the elements of V by the elements of F, such that the following conditions are satisfied:

1) $x \oplus y \in V \quad \forall x, y \in V$.
2) $x \oplus y = y \oplus x \quad \forall x, y \in V$.
3) $x \oplus (y \oplus w) = (x \oplus y) \oplus w \quad \forall x, y, w \in V$.
4) There is an element $0 \in V$ such that $v \oplus 0 = 0 \oplus v = v \quad \forall v \in V$.
5) $\forall v \in V$ there is an element $-v \in V$ such that $v \oplus -v = -v \oplus v = 0$.
6) $cv \in V \quad \forall c \in F$ and $\forall v \in V$.
7) $c(bv) = (c \cdot b)v \quad \forall c, b \in V$.
8) $(c+b)v = cv \oplus bv \quad \forall c, b \in F$ and $\forall v \in V$.
9) $c\,(v \oplus w) = cv \oplus cw$
10) $uv = v \quad \forall v \in V$ and u is the multiplication identity of F.

Definitions:
If V is a vector space over a field $(F,+,\cdot)$ then any element in V is called a vector, the binary operation $\oplus$ defined on V is

12. called vector addition, the elements of F are called scalars, the vector 0 is called the zero vector, the vector $-v$ is called the negative vector.

Remark: Every field is a vector space over itself.

Example 1: The set R (of all real numbers) together with the usual addition of real numbers and the usual multiplication of real numbers is a vector space over the usual field of real numbers $(R, +, \cdot)$.

Example 2: The set C (of all complex numbers) together with the usual addition of complex numbers and the usual multiplication of complex numbers is a vector space over the usual field of complex numbers $(C, +, \cdot)$.

Example 3: The set Q (of all rational numbers) together with the usual addition of rational numbers and the usual multiplication of rational numbers is a vector space over the usual field of rational numbers $(Q, +, \cdot)$.

Example 4: The set $Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$

13. together with $+_7$ (the addition of integers modulo 7) and $\cdot_7$ (the multiplication of integers modulo 7) is a vector space over the field $(\mathbb{Z}_7, +_7, \cdot_7)$.

Remark: Let $(F, +, \cdot)$ be a field. The set $F^n = \{(x_1, x_2, \dots, x_n) \mid x_1, x_2, \dots, x_n \in F\}$ together with the usual addition $+' : F^n \times F^n \longrightarrow F^n$ defined by $(x_1, x_2, \dots, x_n) +' (y_1, y_2, \dots, y_n)$
$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$, $\forall (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \in F^n$ and the usual scalar multiplication $\cdot' : F \times F^n \longrightarrow F^n$ defined by $k \cdot' (x_1, x_2, \dots, x_n) = (k \cdot x_1, k \cdot x_2, \dots, k \cdot x_n)$ $\forall k \in F$ and $\forall (x_1, x_2, \dots, x_n) \in F^n$ is a vector space over the field $(F, +, \cdot)$.

Example 5: Show that $\mathbb{Z}_{13}^3$ together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the field $(\mathbb{Z}_{13}, +_{13}, \cdot_{13})$.

Solution:
$\forall\ v = (v_1, v_2, v_3), w = (w_1, w_2, w_3), y = (y_1, y_2, y_3) \in \mathbb{Z}_{13}^3$ and $\forall\ r, s \in \mathbb{Z}_{13}$, we have:

1) $v + w = (v_1, v_2, v_3) + (w_1, w_2, w_3)$
$= (v_1 +_{13} w_1, v_2 +_{13} w_2, v_3 +_{13} w_3) \in \mathbb{Z}_{13}^3$ since $v_1 +_{13} w_1, v_2 +_{13} w_2, v_3 +_{13} w_3 \in \mathbb{Z}_{13}$,

2) $v + w = (v_1, v_2, v_3) + (w_1, w_2, w_3)$

14.
$= (u_1 +_{13} w_1 , u_2 +_{13} w_2 , u_3 +_{13} w_3)$
$= (w_1 +_{13} u_1 , w_2 +_{13} u_2 , w_3 +_{13} u_3)$
$= (w_1 , w_2 , w_3) + (u_1 , u_2 , u_3) = w + u$,

3) $(u+w)+y$
$= ((u_1 , u_2 , u_3) + (w_1 , w_2 , w_3)) + (y_1 , y_2 , y_3)$
$= (u_1 +_{13} w_1 , u_2 +_{13} w_2 , u_3 +_{13} w_3) + (y_1 , y_2 , y_3)$
$= ((u_1 +_{13} w_1) +_{13} y_1 , (u_2 +_{13} w_2) +_{13} y_2 , (u_3 +_{13} w_3) +_{13} y_3)$
$= (u_1 +_{13} (w_1 +_{13} y_1) , u_2 +_{13} (w_2 +_{13} y_2) , u_3 +_{13} (w_3 +_{13} y_3))$
$= (u_1 , u_2 , u_3) + (w_1 +_{13} y_1 , w_2 +_{13} y_2 , w_3 +_{13} y_3)$
$= (u_1 , u_2 , u_3) + ((w_1 , w_2 , w_3) + (y_1 , y_2 , y_3))$
$= u + (w+y)$,

4) $0 = (0,0,0)$ since
$u + 0 = (u_1 , u_2 , u_3) + (0,0,0)$
$= (u_1 +_{13} 0 , u_2 +_{13} 0 , u_3 +_{13} 0)$
$= (u_1 , u_2 , u_3) = u$,

5) $-u = (-u_1 , -u_2 , -u_3)$ (where $-u_i = 13 - u_i$ when $u_i \neq 0$ while $-0 = 0$) since
$u + (-u) = (u_1 , u_2 , u_3) + (-u_1 , -u_2 , -u_3)$
$= (u_1 +_{13} -u_1 , u_2 +_{13} -u_2 , u_3 +_{13} -u_3)$
$= (0,0,0)$,

6) $r \cdot u = r \cdot (u_1 , u_2 , u_3)$
$= (r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot_{13} u_3) \in \mathbb{Z}_{13}^3$
since $r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot u_3 \in \mathbb{Z}_{13}$,

7) $r \cdot (s \cdot u) = r \cdot (s \cdot (u_1 , u_2 , u_3))$
$= r \cdot (s \cdot_{13} u_1 , s \cdot_{13} u_2 , s \cdot_{13} u_3)$
$= (r \cdot_{13} (s \cdot_{13} u_1) , r \cdot_{13} (s \cdot_{13} u_2) , r \cdot_{13} (s \cdot_{13} u_3))$

15. $= ((r \cdot_{13} s) \cdot_{13} u_1 , (r \cdot_{13} s) \cdot_{13} u_2 , (r \cdot_{13} s) \cdot_{13} u_3)$

$= (r \cdot_{13} s) \cdot (u_1 , u_2 , u_3)$

$= (r \cdot_{13} s) \cdot u$,

8) $(r +_{13} s) \cdot u$

$= (r +_{13} s) \cdot (u_1 , u_2 , u_3)$

$= ((r +_{13} s) \cdot_{13} u_1 , (r +_{13} s) \cdot_{13} u_2 , (r +_{13} s) \cdot_{13} u_3)$

$= ((r \cdot_{13} u_1) +_{13} (s \cdot_{13} u_1) , (r \cdot_{13} u_2) +_{13} (s \cdot_{13} u_2) , (r \cdot_{13} u_3) +_{13} (s \cdot_{13} u_3))$

$= (r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot_{13} u_3) + (s \cdot_{13} u_1 , s \cdot_{13} u_2 , s \cdot_{13} u_3)$

$= r \cdot (u_1 , u_2 , u_3) + s \cdot (u_1 , u_2 , u_3)$

$= r \cdot u + s \cdot u$,

9) $r \cdot (u + w)$

$= r \cdot ((u_1 , u_2 , u_3) + (w_1 , w_2 , w_3))$

$= r \cdot (u_1 +_{13} w_1 , u_2 +_{13} w_2 , u_3 +_{13} w_3)$

$= (r \cdot_{13} (u_1 +_{13} w_1) , r \cdot_{13} (u_2 +_{13} w_2) , r \cdot_{13} (u_3 +_{13} w_3))$

$= ((r \cdot_{13} u_1) +_{13} (r \cdot_{13} w_1) , (r \cdot_{13} u_2) +_{13} (r \cdot_{13} w_2) , (r \cdot_{13} u_3) +_{13} (r \cdot_{13} w_3))$

$= (r \cdot_{13} u_1 , r \cdot_{13} u_2 , r \cdot_{13} u_3) + (r \cdot_{13} w_1 , r \cdot_{13} w_2 , r \cdot_{13} w_3)$

$= r \cdot u_1 + r \cdot u_2$,

10) $1 \cdot u$

$= 1 \cdot (u_1 , u_2 , u_3)$

$= (1 \cdot_{13} u_1 , 1 \cdot_{13} u_2 , 1 \cdot_{13} u_3)$

$= (u_1 , u_2 , u_3) = u$.

Thus $\mathbb{Z}_{13}^{3}$ **together with the** usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the Field $(\mathbb{Z}_{13} , +_{13} , \cdot_{13})$.

16. Example 6 : Show that Q^4 together with the usual addition of vectors + and the usual scalar multiplication $\cdot$ is a vector space over the usual field of rational numbers $(Q, +, \cdot)$.

Solution :

$\forall\, \boldsymbol{v} = (v_1, v_2, v_3, v_4)$, $\boldsymbol{w} = (w_1, w_2, w_3, w_4)$,
$\boldsymbol{y} = (y_1, y_2, y_3, y_4) \in Q^4$ and $\forall\, r, s \in Q$, we have:

1) $\boldsymbol{v} + \boldsymbol{w} = (v_1, v_2, v_3, v_4) + (w_1, w_2, w_3, w_4)$
$= (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4) \in Q^4$
since $v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4 \in Q$,

2) $\boldsymbol{v} + \boldsymbol{w} = (v_1, v_2, v_3, v_4) + (w_1, w_2, w_3, w_4)$
$= (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4)$
$= (w_1 + v_1, w_2 + v_2, w_3 + v_3, w_4 + v_4)$
$= (w_1, w_2, w_3, w_4) + (v_1, v_2, v_3, v_4) = \boldsymbol{w} + \boldsymbol{v}$,

3) $(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{y}$
$= ((v_1, v_2, v_3, v_4) + (w_1, w_2, w_3, w_4)) + (y_1, y_2, y_3, y_4)$
$= (v_1 + w_1, v_2 + w_2, v_3 + w_3, v_4 + w_4) + (y_1, y_2, y_3, y_4)$
$= ((v_1 + w_1) + y_1, (v_2 + w_2) + y_2, (v_3 + w_3) + y_3, (v_4 + w_4) + y_4)$
$= (v_1 + (w_1 + y_1), v_2 + (w_2 + y_2), v_3 + (w_3 + y_3), v_4 + (w_4 + y_4))$
$= (v_1, v_2, v_3, v_4) + (w_1 + y_1, w_2 + y_2, w_3 + y_3, w_4 + y_4)$
$= (v_1, v_2, v_3, v_4) + ((w_1, w_2, w_3, w_4) + (y_1, y_2, y_3, y_4))$
$= \boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{y})$,

4) $\boldsymbol{0} = (0, 0, 0, 0)$ since
$\boldsymbol{v} + \boldsymbol{0} = (v_1, v_2, v_3, v_4) + (0, 0, 0, 0)$

17. 5) $-U = (-U_1, -U_2, -U_3, -U_4)$ since

$$U + (-U) = (U_1, U_2, U_3, U_4) + (-U_1, -U_2, -U_3, -U_4)$$
$$= (U_1 - U_1, U_2 - U_2, U_3 - U_3, U_4 - U_4)$$
$$= (0, 0, 0, 0) = \mathbf{0} \quad ,$$

6) $r \cdot U = r \cdot (U_1, U_2, U_3, U_4)$
$$= (r \cdot U_1, r \cdot U_2, r \cdot U_3, r \cdot U_4) \in \mathbb{Q}^4 \quad \text{since}$$
$r \cdot U_1, r \cdot U_2, r \cdot U_3, r \cdot U_4 \in \mathbb{Q}$,

7) $r \cdot (s \cdot U) = r \cdot (s \cdot (U_1, U_2, U_3, U_4))$
$$= r \cdot (s \cdot U_1, s \cdot U_2, s \cdot U_3, s \cdot U_4)$$
$$= (r \cdot (s \cdot U_1), r \cdot (s \cdot U_2), r \cdot (s \cdot U_3), r \cdot (s \cdot U_4))$$
$$= ((r \cdot s) \cdot U_1, (r \cdot s) \cdot U_2, (r \cdot s) \cdot U_3, (r \cdot s) \cdot U_4)$$
$$= (r \cdot s) \cdot (U_1, U_2, U_3, U_4)$$
$$= (r \cdot s) \cdot U$$

8) $(r+s) \cdot U$
$$= (r+s) \cdot (U_1, U_2, U_3, U_4)$$
$$= ((r+s) \cdot U_1, (r+s) \cdot U_2, (r+s) \cdot U_3, (r+s) \cdot U_4)$$
$$= (r \cdot U_1 + s \cdot U_1, r \cdot U_2 + s \cdot U_2, r \cdot U_3 + s \cdot U_3, r \cdot U_4 + s \cdot U_4)$$
$$= (r \cdot U_1, r \cdot U_2, r \cdot U_3, r \cdot U_4) + (s \cdot U_1, s \cdot U_2, s \cdot U_3, s \cdot U_4)$$
$$= r \cdot (U_1, U_2, U_3, U_4) + s \cdot (U_1, U_2, U_3, U_4)$$
$$= r \cdot U + s \cdot U$$

9) $r \cdot (U + W)$
$$= r \cdot ((U_1, U_2, U_3, U_4) + (W_1, W_2, W_3, W_4))$$
$$= r \cdot (U_1 + W_1, U_2 + W_2, U_3 + W_3, U_4 + W_4)$$
$$= (r \cdot (U_1 + W_1), r \cdot (U_2 + W_2), r \cdot (U_3 + W_3), r \cdot (U_4 + W_4))$$
$$= (r \cdot U_1 + r \cdot W_1, r \cdot U_2 + r \cdot W_2, r \cdot U_3 + r \cdot W_3, r \cdot U_4 + r \cdot W_4)$$
$$= (r \cdot U_1, r \cdot U_2, r \cdot U_3, r \cdot U_4) + (r \cdot W_1, r \cdot W_2, r \cdot W_3, r \cdot W_4)$$

18.

$= r \cdot (u_1, u_2, u_3, u_4) + r \cdot (w_1, w_2, w_3, w_4)$

$= r \cdot U + r \cdot W$

10) $1 \cdot U$

$= 1 \cdot (u_1, u_2, u_3, u_4)$

$= (1 \cdot u_1, 1 \cdot u_2, 1 \cdot u_3, 1 \cdot u_4)$

$= (u_1, u_2, u_3, u_4)$

$= U$

Thus Q^4 together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the usual field of rational numbers $(Q, +, \cdot)$.

Remark: Let $(F, +, \cdot)$ be a field. The set $V = M_{m \times n}(F)$ of all $m \times n$ matrices over the field $(F, +, \cdot)$ (i.e. the entries of the matrices belong to F) together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the field $(F, +, \cdot)$.

Example 7: Show that $M_{2 \times 2}(R)$ together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the