linear Algebra 1

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Lecture 3

Vector space

18. $= r\cdot(u_1, u_2, u_3, u_4) + r\cdot(w_1, w_2, w_3, w_4)$

$= r\cdot U + r\cdot W$

10) $1\cdot U$

$= 1\cdot(u_1, u_2, u_3, u_4)$

$= (1\cdot u_1, 1\cdot u_2, 1\cdot u_3, 1\cdot u_4)$

$= (u_1, u_2, u_3, u_4)$

$= U$

Thus Q^4 together with the usual addition of vectors $+$ and the usual scalar multiplication $\cdot$ is a vector space over the usual field of rational numbers $(Q, +, \cdot)$.

Remark: Let $(F, +, \cdot)$ be a field. The set $V = M_{m\times n}(F)$ of all $m\times n$ matrices over the field $(F, +, \cdot)$ (i.e. the entries of the matrices belong to F) together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the field $(F, +, \cdot)$.

Example 7: Show that $M_{2\times 2}(R)$ together with the usual addition of matrices and the usual scalar multiplication of matrices by scalars is a vector space over the

19. usual field of real numbers $(R, +, \cdot)$.

Solution:

$\forall\, v = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}, w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}, y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \in M_{2\times2}(R)$

and $\forall\, r, s \in R$, we have

1) $$v + w = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{pmatrix}$$

$\in M_{2\times2}(R)$ since $v_{11}+w_{11}, v_{12}+w_{12}, v_{21}+w_{21}, v_{22}+w_{22} \in R$,

2) $$\begin{aligned} v + w &= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \\ &= \begin{pmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{pmatrix} \\ &= \begin{pmatrix} w_{11}+v_{11} & w_{12}+v_{12} \\ w_{21}+v_{21} & w_{22}+v_{22} \end{pmatrix} \\ &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \\ &= w + v, \end{aligned}$$

3) $(v + w) + y$

$$\begin{aligned} &= \left(\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} \right) + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \\ &= \begin{pmatrix} (v_{11}+w_{11})+y_{11} & (v_{12}+w_{12})+y_{12} \\ (v_{21}+w_{21})+y_{21} & (v_{22}+w_{22})+y_{22} \end{pmatrix} \end{aligned}$$

20.

$$= \begin{pmatrix} u_{11}+(w_{11}+y_{11}) & u_{12}+(w_{12}+y_{12}) \\ u_{21}+(w_{21}+y_{21}) & u_{22}+(w_{22}+y_{22}) \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} w_{11}+y_{11} & w_{12}+y_{12} \\ w_{21}+y_{21} & w_{22}+y_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \left(\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} + \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \right)$$

$$= U + (W + Y) \quad ,$$

4) $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ since

$$U + 0 = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u_{11}+0 & u_{12}+0 \\ u_{21}+0 & u_{22}+0 \end{pmatrix}$$

$$= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = U \quad ,$$

5) $-U = \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$ since

$$U + (-U) = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} u_{11}-u_{11} & u_{12}-u_{12} \\ u_{21}-u_{21} & u_{22}-u_{22} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad ,$$

6) $r \circ U = r \circ \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} r\cdot u_{11} & r\cdot u_{12} \\ r\cdot u_{21} & r\cdot u_{22} \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$

21. since $r\cdot u_{11}, r\cdot u_{12}, r\cdot u_{21}, r\cdot u_{22} \in R$,

7) $$r\cdot(s\bullet U) = r\cdot\left(s\bullet\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix}\right)$$

$$= r\cdot\begin{pmatrix} s\cdot u_{11} & s\cdot u_{12} \\ s\cdot u_{21} & s\cdot u_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot(s\cdot u_{11}) & r\cdot(s\cdot u_{12}) \\ r\cdot(s\cdot u_{21}) & r\cdot(s\cdot u_{22})\end{pmatrix}$$

$$= \begin{pmatrix} (r\cdot s)\cdot u_{11} & (r\cdot s)\cdot u_{12} \\ (r\cdot s)\cdot u_{21} & (r\cdot s)\cdot u_{22}\end{pmatrix}$$

$$= (r\cdot s)\bullet\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix}$$

$$= (r\cdot s)\bullet U \quad ,$$

8.) $(r+s)\bullet U$

$$= (r+s)\bullet\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix}$$

$$= \begin{pmatrix} (r+s)\cdot u_{11} & (r+s)\cdot u_{12} \\ (r+s)\cdot u_{21} & (r+s)\cdot u_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot u_{11}+s\cdot u_{11} & r\cdot u_{12}+s\cdot u_{12} \\ r\cdot u_{21}+s\cdot u_{21} & r\cdot u_{22}+s\cdot u_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot u_{11} & r\cdot u_{12} \\ r\cdot u_{21} & r\cdot u_{22}\end{pmatrix} + \begin{pmatrix} s\cdot u_{11} & s\cdot u_{12} \\ s\cdot u_{21} & s\cdot u_{22}\end{pmatrix}$$

$$= r\bullet\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix} + s\bullet\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix} = r\cdot U + r\cdot s \quad ,$$

22. 9) $r\cdot(v+w)$

$$= r\cdot\left(\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} + \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix}\right)$$

$$= r\cdot\begin{pmatrix} v_{11}+w_{11} & v_{12}+w_{12} \\ v_{21}+w_{21} & v_{22}+w_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot(v_{11}+w_{11}) & r\cdot(v_{12}+w_{12}) \\ r\cdot(v_{21}+w_{21}) & r\cdot(v_{22}+w_{22})\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot v_{11}+r\cdot w_{11} & r\cdot v_{12}+r\cdot w_{12} \\ r\cdot v_{21}+r\cdot w_{21} & r\cdot v_{22}+r\cdot w_{22}\end{pmatrix}$$

$$= \begin{pmatrix} r\cdot v_{11} & r\cdot v_{12} \\ r\cdot v_{21} & r\cdot v_{22}\end{pmatrix} + \begin{pmatrix} r\cdot w_{11} & r\cdot w_{12} \\ r\cdot w_{21} & r\cdot w_{22}\end{pmatrix}$$

$$= r\cdot\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} + r\cdot\begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22}\end{pmatrix}$$

$$= r\cdot v + r\cdot w$$

10) $1\cdot v = 1\cdot\begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22}\end{pmatrix} = v$

Thus $M_{2\times2}(R)$ together with the usual addition of matrices and the usual scalar multiplication of matrices by scalar is a vector space over the usual field of real numbers $(R,+,\cdot)$.

<u>Remark</u>: Let $(F,+,\cdot)$ be a field. The set $F[x]$ of all polynomials in x over the field

23. $(F, +, \cdot)$ (i.e. $F[x]$ is the set of all functions that can be expressed as $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_0, a_1, \dots, a_n \in F$) is a vector space over the field $(F, +, \cdot)$.

Remarks:

1) Every vector space V over the usual field of complex numbers is also a vector space over the usual field of real numbers and also a vector space over the usual field of rational numbers.
2) Every vector space V over the usual field of real numbers is also a vector space over the usual field of rational numbers.

Theorem 1: Let the set V together with the binary operation $+$ and the scalar multiplication $\cdot$ (multiplication of vectors by scalars) be a vector space over a field $(F, +, \cdot)$, then:

i) The identity element (with respect to $+$) is unique.

ii) Each element v of V has exactly one inverse element (with respect to $+$).

iii) If $v, w, y \in V$ and $y + v = y + w$ then $v = w$.

iv) If $v, w, y \in V$ and $v + y = w + y$ then $v = w$.

v) $z \cdot v = 0 \quad \forall v \in V$.

24. vi) $k \cdot 0 = 0 \quad \forall k \in F$.

vii) If $k \cdot v = 0$ then $k = z$ or $v = 0$.

viii) $(-u) \cdot v = -v \quad \forall v \in V$.

Proof:

i) Assume that V has two identity elements 0 and e, then we have

$$\left.\begin{array}{l} v + 0 = 0 + v = v \\ \text{and } v + e = e + v = v \end{array}\right\} \forall v \in V.$$

Hence $e = 0 + e = 0$.

Thus the identity element is unique

ii) Let $v \in V$ and assume that v has two inverses $-v$ and w, then we have

$$v + -v = -v + v = 0$$

and $v + w = w + v = 0$

Hence

$$\begin{aligned} -v &= -v + 0 \\ &= -v + (v + w) \\ &= (-v + v) + w \\ &= 0 + w \\ &= w. \end{aligned}$$

Thus v has only one inverse element.

iii) If $y + v = y + w$, then

$$\begin{aligned} v &= 0 + v \\ &= (-y + y) + v \end{aligned}$$

25.

$= -y + (y + v)$
$= -y + (y + w)$
$= (-y + y) + w$
$= 0 + w$
$= w$.

iv) If $v + y = w + y$, then
$v = v + 0$
$= v + (y + -y)$
$= (v + y) + -y$
$= (w + y) + -y$
$= w + (y + -y)$
$= w + 0$
$= w$

v) $0 + (z \cdot v)$
$= z \cdot v$
$= (z + z) \cdot v$
$= (z \cdot v) + (z \cdot v) \quad \forall v \in V$
Then by (iv) above $0 = z \cdot v \quad \forall v \in V$.

vi) $0 + (k \cdot 0)$
$= k \cdot 0$
$= k \cdot (0 + 0)$
$= (k \cdot 0) + (k \cdot 0) \quad \forall k \in F$
Then by (iv) above $0 = k \cdot 0 \quad \forall k \in F$

vii) By (v) above $k \bullet v = 0$ when $k = z$.
Assume that $k \bullet v = 0$ and $k \neq z$, then

$$0 = \frac{1}{k} \bullet 0 \quad \text{by (vi) above}$$

$$= \frac{1}{k} \bullet (k \bullet v)$$

$$= \left(\frac{1}{k} \cdot k\right) \bullet v$$

$$= u \bullet v$$

$$= v$$

viii) $(-u) \bullet v$

$$= (-u) \bullet v + 0$$

$$= (-u) \bullet v + (v + -v)$$

$$= ((-u) \bullet v + v) + -v$$

$$= ((-u) \bullet v + u \bullet v) + -v$$

$$= (-u + u) \bullet v + -v$$

$$= z \bullet v + -v$$

$$= 0 + -v$$

$$= -v$$

§4: Subspaces

Definition: Let V together with a binary operation $+$ defined on V and a scalar