linear Algebra 1

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Lecture 4

Subspace

vii) By (v) above $k \bullet v = 0$ when $k = z$

Assume that $k \bullet v = 0$ and $k \neq z$, then

$$0 = \frac{1}{k} \bullet 0 \qquad \text{by (vi) above}$$
$$= \frac{1}{k} \bullet (k \bullet v)$$
$$= \left(\frac{1}{k} \cdot k\right) \bullet v$$
$$= u \bullet v$$
$$= v$$

viii) $(-u) \bullet v$

$$= (-u) \bullet v + 0$$
$$= (-u) \bullet v + (v + -v)$$
$$= ((-u) \bullet v + v) + -v$$
$$= ((-u) \bullet v + u \bullet v) + -v$$
$$= (-u + u) \bullet v + -v$$
$$= z \bullet v + -v$$
$$= 0 + -v$$
$$= -v$$

§4: Subspaces

Definition: Let V together with a binary operation $+$ defined on V and a scalar

27. multiplication $\cdot$ be a vector space over a field $(F,+,\cdot)$ and let W be a nonempty subset of V. If W is a vector space over the same field $(F,+,\cdot)$ with respect to the same operations $+$ and $\cdot$ of the vector space V, then W is called a subspace of V.

<u>Theorem 2:</u> W is a subspace of a vector space V over a field $(F,+,\cdot)$ iff

i) W is a nonempty subset of V (i.e. $\phi \neq W \subseteq V$).

ii) W is closed under the binary operation $+$ defined on V (i.e. $v + w \in W \quad \forall v, w \in W$).

iii) W is closed under the scalar multiplication $\cdot$ defined on $F \times V$ (i.e. $k \cdot v \in W \quad \forall k \in F$ and $\forall v \in W$)

<u>Proof:</u> If W is a subspace of V then by the definition of subspace $W \subseteq V$ and W is a vector space under the binary operation $+$ defined on V and the scalar multiplication $\cdot$ defined on $F \times V$ (i.e. $\cdot : F \times V \longrightarrow V$). Hence by the definition of vector space W is nonempty which is a subset of V (i.e. (i) is satisfied), and

28. $v + w \in W$ $\forall v, w \in W$ by the first property of the definition of vector space (i.e. (ii) is satisfied) and $k \cdot v \in W$ $\forall k \in F$ and $\forall v \in W$ by the sixth property of the definition of vector space (i.e. (iii) is satisfied).

Conversely, if W satisfies (i), (ii), and (iii) then $W \neq \phi$ and

1) $\forall v, w$ we have $v + w \in W$ by (i).
2) $\forall v, w \in W$ we have $v, w \in V$ (since $W \subseteq V$) and hence $v + w = w + v$.
3) $\forall v, w, y \in W$ we have $v, w, y \in V$ (since $W \subseteq V$) and hence $v + (w + y) = (v + w) + y$.
4) Since $W \neq \phi$ then there exists $w \in W$. Hence by (iii) $z \cdot w = 0 \in W$ and $\forall v \in W$ we have $v \in V$ (since $W \subseteq V$) and thus $v + 0 = v$.
5) $\forall v \in W$ we have by (iii) $-u \cdot v = -v$ and $v + -v = 0$.
6) $\forall k \in F$ and $\forall v \in W$ we have $k \cdot v \in W$ by (iii).
7) $\forall k \in F$ and $\forall v, w \in W$ we have $v, w \in V$, hence $k \cdot (v + w) = k \cdot v + k \cdot w$.
8) $\forall k, l \in F$ and $\forall v \in W$ we have $v \in V$, hence $(k + l) \cdot v = k \cdot v + l \cdot v$.
9) $\forall k, l \in F$ and $\forall v \in W$ we have $v \in V$, hence $(k \cdot l) \cdot v = k \cdot (l \cdot v)$.

29. 10) $\forall v \in W$ we have $v \in V$, hence $u \cdot v = v$.

Example 1: List all the subspaces of the vector space R^2 over $(R,+,\cdot)$.
Solution: The subspaces of the vector space R^2 over $(R,+,\cdot)$ are:
1) $\{(0,0)\}$ is a subspace.
2) $\forall k,l \in R \ni (k,l) \neq (0,0)$ we have $W_{k,l} = \{(kt,lt) \mid t \in R\}$ is a subspace.
(i.e. every straight line passing through the origin $O(0,0)$ of the plane R^2 is a subspace of R^2 over $(R,+,\cdot)$).
3) R^2 is a subspace.

Example 2: List all the subspaces of the vector space R^3 over $(R,+,\cdot)$.
Solution: The subspaces of the vector space R^3 over $(R,+,\cdot)$ are:
1) $\{(0,0,0)\}$ is a subspace.
2) $\forall k,l,m \in R \ni (k,l,m) \neq (0,0,0)$ we have $W_{k,l,m} = \{(kt,lt,mt) \mid t \in R\}$ is a subspace. (i.e. every straight line passing through the origin $O(0,0,0)$ of the space R^3 over $(R,+,\cdot)$).
3) $\forall k,l,m \in R \ni (k,l,m) \neq (0,0,0)$ we

30. have $V_{k,l,m} = \{(x,y,z) \in R^3 \mid kx+lm+mz=0\}$ is a subspace. (i.e. every plane passing through the origin $O(0,0,0)$ of the space R^3 is a subspace R^3 over $(R,+,\cdot)$).

4) R^3 is a subspace.

Example 3: Show that $W=\{(x,0,z) \mid x,z \in R\}$ is a subspace of the vector space R^3 over $(R,+,\cdot)$.

Solution: The vector $(0,0,0) \in W$ since the second component is 0. Thus $W \neq \phi$.

$\forall\, v=(v_1,0,v_3),\ w=(w_1,0,w_3) \in W$ we have

$v+w=(v_1,0,v_3)+(w_1,0,w_3)$

$\quad =(v_1+w_1,0,v_3+w_3) \in W$ since v_1+w_1, $v_3+w_3 \in R$ and the second component is 0.

$\forall k \in R$ and $\forall v=(v_1,0,v_3) \in W$ we have

$k\cdot v = k\cdot(v_1,0,v_3)=(kv_1,0,kv_3) \in W$ since $kv_1, kv_3 \in R$ and the second compotent is 0.

Thus by theorem 2, W is a subspace of R^3 over $(R,+,\cdot)$.

Example 4: Show that $W=\{(x,y,z) \mid x+y+2z=0\}$ is a subspace of the vector space R^3 over $(R,+,\cdot)$.

Solution: The vector $(0,0,0) \in W$ since

$0+0+2\times 0=0$. Thus $W\neq \phi$.

$\forall\, v=(v_1,v_2,v_3)$, $w=(w_1,w_2,w_3)\in W$ we have

$v+w=(v_1,v_2,v_3)+(w_1,w_2,w_3)$

$\quad =(v_1+w_1, v_2+w_2, v_3+w_3)\in W$ since

$(v_1+w_1)+(v_2+w_2)+2(v_3+w_3)$

$=v_1+w_1+v_2+w_2+2v_3+2w_3$

$=v_1+v_2+2v_3+w_1+w_2+2w_3$

$=0+0$ since $v=(v_1,v_2,v_3), w=(w_1,w_2,w_3)$ $\in W$ which means that $v_1+v_2+2v_3=0$, $w_1+w_2+2w_3=0$

$=0$

$\forall k\in R$ and $\forall v=(v_1,v_2,v_3)\in W$ we have

$k\cdot v=k\cdot(v_1,v_2,v_3)$

$\quad =(kv_1, kv_2, kv_3)\in W$ since

$kv_1+kv_2+2kv_3$

$=k(v_1+v_2+2v_3)$

$=k\cdot 0$ since $(v_1,v_2,v_3)\in W$ which means that $v_1+v_2+2v_3=0$

$=0$

Thus by theorem 2 , W is a subspace of R^3 over $(R,+,\cdot)$.

Example 5 : Show that $W=\left\{\begin{pmatrix} t & s \\ 3t & 5t\end{pmatrix} \middle| t,s\in R\right\}$ is a subspace of the vector space $M_{2\times 2}(R)$ over $(R,+,\cdot)$.

Solution: The matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\in W$. Thus $W\neq\phi$.

32- $\forall \begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix}, \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} \in W$ we have

$$\begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix} + \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} = \begin{pmatrix} t_1+t_2 & s_1+s_2 \\ 3t_1+3t_2 & 5t_1+5t_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1+t_2 & s_1+s_2 \\ 3(t_1+t_2) & 5(t_1+t_2) \end{pmatrix} \in W \quad \text{since } t_1+t_2, s_1+s_2 \in R.$$

$\forall k \in R$ and $\forall \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} \in W$ we have

$$k \cdot \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} = \begin{pmatrix} k.t & ks \\ k.3t & k.5t \end{pmatrix}$$

$$= \begin{pmatrix} kt & ks \\ 3kt & 5kt \end{pmatrix} \in W \quad \text{since } kt, ks \in R.$$

§5 : Bases and dimension

Definition: Let V be a vector space over a field $(F, +, \cdot)$. A vector w is called a linear combination of the vectors $v_1, v_2, \ldots, v_m$ if $w = k_1 v_1 + k_2 v_2 + \cdots + k_m v_m$ for some $k_1, k_2, \ldots, k_m \in F$.

Example 1: In the vector space R^3 over $(R, +, \cdot)$ we have $w = (2, 6, 1)$ is a linear combination of $v_1 = (1,1,0)$, $v_2 = (2,0,3)$, and $v_3 = (1,3,4)$ since

$3v_1 - v_2 + v_3 = 3(1,1,1) - (2,0,3) + (1,3,4)$