linear Algebra 1

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Lecture 5

Linear dependence

32. $\forall \begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix}, \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} \in W$ we have

$$\begin{pmatrix} t_1 & s_1 \\ 3t_1 & 5t_1 \end{pmatrix} + \begin{pmatrix} t_2 & s_2 \\ 3t_2 & 5t_2 \end{pmatrix} = \begin{pmatrix} t_1+t_2 & s_1+s_2 \\ 3t_1+3t_2 & 5t_1+5t_2 \end{pmatrix}$$

$$= \begin{pmatrix} t_1+t_2 & s_1+s_2 \\ 3(t_1+t_2) & 5(t_1+t_2) \end{pmatrix} \in W \quad \text{since } t_1+t_2, s_1+s_2 \in R.$$

$\forall k \in R$ and $\forall \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} \in W$ we have

$$k \cdot \begin{pmatrix} t & s \\ 3t & 5t \end{pmatrix} = \begin{pmatrix} k.t & ks \\ k.3t & k.5t \end{pmatrix}$$

$$= \begin{pmatrix} kt & ks \\ 3kt & 5kt \end{pmatrix} \in W \quad \text{since } kt, ks \in R.$$

§ 5 : Bases and dimension

Definition: Let V be a vector space over a field $(F, +, \cdot)$. A vector w is called a linear combination of the vectors $v_1, v_2, \ldots, v_m$ if $w = k_1 v_1 + k_2 v_2 + \cdots + k_m v_m$ for some $k_1, k_2, \ldots, k_m \in F$.

Example 1: In the vector space R^3 over $(R, +, \cdot)$ we have $w = (2, 6, 1)$ is a linear combination of $v_1 = (1, 1, 0)$, $v_2 = (2, 0, 3)$, and $v_3 = (1, 3, 4)$ since

$3v_1 - v_2 + v_3 = 3(1,1,1) - (2,0,3) + (1,3,4)$

$= (3,3,0) - (2,0,3) + (1,3,4)$
$= (3-2+1\ , 3-0+3\ , 0-3+4)$
$= (2,6,1)$.

Example 2: In the vector space R^3 over $(R,+,\cdot)$, show that $w=(5,7,-1)$ is a linear combination of the vectors $v_1=(1,1,1)$, $v_2=(1,0,1)$, and $v_3=(2,1,3)$.

Solution: To show that w is a linear combination of v_1, v_2, v_3 we should find $k_1, k_2, k_3 \in R$ such that $k_1v_1+k_2v_2+k_3v_3=w$.

Now $k_1v_1+k_2v_2+k_3v_3=w$

$\Rightarrow k_1(1,1,1)+k_2(1,0,1)+k_3(2,1,3)=(5,7,-1)$

$\Rightarrow (k_1,k_1,k_1)+(k_2,0,k_2)+(2k_3,k_3,3k_3)=(5,7,-1)$

$\Rightarrow (k_1+k_2+2k_3\ , k_1+0+k_3\ , k_1+k_2+3k_3)=(5,7,-1)$

$$\Rightarrow \begin{aligned} k_1+k_2+2k_3 &= 5 \\ k_1 \quad\ \ + k_3 &= 7 \\ k_1+k_2+3k_3 &= -1 \end{aligned}$$

by using Cramer's rule we have

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 3 \end{vmatrix} = 1\cdot\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1\cdot(-1) - 1\cdot(2) + 2\cdot(1) = -1-2+2 = -1 .$$

$$|A_1| = \begin{vmatrix} 5 & 1 & 2 \\ 7 & 0 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 5\cdot\begin{vmatrix} 0 & 1 \\ 1 & 3 \end{vmatrix} - 1\cdot\begin{vmatrix} 7 & 1 \\ -1 & 3 \end{vmatrix} + 2\cdot\begin{vmatrix} 7 & 0 \\ -1 & 1 \end{vmatrix}$$

$$= 5\cdot(-1) - 1\cdot(22) + 2\cdot(7) = -5 - 22 + 14 = -13$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 2 \\ 1 & 7 & 1 \\ 1 & -1 & 3 \end{vmatrix} = 1\cdot\begin{vmatrix} 7 & 1 \\ -1 & 3 \end{vmatrix} - 5\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix}$$

$$= 1\cdot(22) - 5(2) + 2(-8) = 22 - 10 - 16 = -4$$

$$|A_3| = \begin{vmatrix} 1 & 1 & 5 \\ 1 & 0 & 7 \\ 1 & 1 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} 0 & 7 \\ 1 & -1 \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix} + 5\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 1\cdot(-7) - 1\cdot(-8) + 5\cdot(1) = -7 + 8 + 5 = 6.$$

Thus $k_1 = \dfrac{|A_1|}{|A|} = \dfrac{-13}{-1} = 13$, $k_2 = \dfrac{|A_2|}{|A|} = \dfrac{-4}{-1} = 4$,

and $k_3 = \dfrac{|A_3|}{|A|} = \dfrac{6}{-1} = -6$.

Hence $w = 13u_1 + 4u_2 - 6u_3$.

Definition: Let V be a vector space over a field $(F, +, \cdot)$ and let $S = \{u_1, u_2, \dots, u_m\} \subseteq V$, then the subspace W of V consisting of all linear combinations of the vectors in S is called the subspace (or the vector space) spanned by $u_1, u_2, \dots, u_m$ and we say that the vectors $u_1, u_2, \dots, u_m$ span W, and we denoted it by $W = \text{span}(S)$ or $W = \text{span}\{u_1, u_2, \dots, u_m\}$.

35. **Example**: Let $W=\{(x,y,0) \mid x,y \in R\}$ is a subspace of R^3 over $(R,+,\cdot)$. Show that $W=\text{span}(S)$, where $S=\{(1,0,0),(0,1,0)\}$.

Solution: Let $v=(x_1,y_1,0)$ be any vector belongs to W then

$v=(x_1,y_1,0)=x_1\cdot(1,0,0)+y_1(0,1,0)$.

Thus every vector belongs to W is a linear combination of the vectors in S.

Therefore $W=\text{span}(S)$.

Definition: If $S=\{v_1,v_2,\dots,v_m\}$ is a nonempty set of vectors, then the vector equation

$k_1v_1+k_2v_2+\dots+k_mv_m=0$

has at least one solution which is

$k_1=k_2=\dots=k_m=z$.

If this is the only solution, then S is called a linearly independent set. If there are other solutions, then S is called a linearly dependent set.

Theorem 3: Let V be a vector space over a field $(F,+,\cdot)$.

i- A subset S of V with more than one element is linearly dependent iff at least one of the vectors is a linear combination of the other vectors in S.

ii- A subset S of V with more than one element is linearly independent iff no vector in S is a linear combination of the other vectors in S.

iii- A finite subset S of V that contains the zero vector 0 is linearly dependent.

iv- A subset S of V with exactly two vectors is linearly independent iff neither vector is a scalar multiple of the other.

Example 1: If $v_1 = (3, 1, 5)$ and $v_2 = (9, 3, 15)$ then the set $S = \{v_1, v_2\}$ in the vector space R^3 over the field $(R, +, \cdot)$ is linearly dependent since $v_2 = 3v_1$.

Example 2: If $v_1 = (2, 3, 4)$, $v_2 = (1, 1, 2)$ and $v_3 = (3, 5, 6)$ then the set $S = \{v_1, v_2, v_3\}$ in the vector space R^3 over the field $(R, +, \cdot)$ is linearly dependent since $2v_1 - v_2 - v_3 = (0, 0, 0) = 0$.

Example 3: Let $Q[x]$ be the set of all polynomials in x with rational coefficients.

37. $Q[x]$ is a vector space over the field $(Q, +, \cdot)$

i- Prove or disprove that the set $S = \{p_1(x), p_2(x), p_3(x)\}$ which consists of the polynomials $p_1(x) = 1 - x$, $p_2(x) = 2 + 3x - 2x^2$ and $p_3(x) = -\frac{1}{2} + 3x - x^2$ is linearly dependent.

ii- Prove or disprove that the set $T = \{p_1(x), p_2(x)\}$ where $p_1(x)$ and $p_2(x)$ are the same polynomials given in (i) above is linearly dependent.

Solution:

i- $k_1 p_1(x) + k_2 p_2(x) + k_3 p_3(x) = 0$

$$\Rightarrow k_1(1-x) + k_2(2+3x-2x^2) + k_3(-\tfrac{1}{2} + 3x - x^2) = 0$$

$$\Rightarrow k_1 - k_1 x + 2k_2 + 3k_2 x - 2k_2 x^2 - \tfrac{1}{2}k_3 + 3k_3 x - k_3 x^2 = 0$$

$$\Rightarrow (k_1 + 2k_2 - \tfrac{1}{2}k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

$$\Rightarrow \quad k_1 + 2k_2 - \tfrac{1}{2}k_3 = 0 \qquad \text{---} (1)$$

$$-k_1 + 3k_2 + 3k_3 = 0 \qquad \text{---} (2)$$

$$-2k_2 - k_3 = 0 \qquad \text{---} (3)$$

$$\Rightarrow |A| = \begin{vmatrix} 1 & 2 & -\frac{1}{2} \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} - \tfrac{1}{2}\begin{vmatrix} -1 & 3 \\ 0 & -2 \end{vmatrix}$$

$$= 1 \cdot (3) - 2 \cdot (1) - \tfrac{1}{2}(2) = 3 - 2 - 1 = 0$$

which means that we should take only two of the equations (1), (2), and (3) to find the values of k_1, k_2, and k_3, we will take

38- ① and ②

$\Rightarrow \quad k_1 + 2k_2 = \frac{1}{2} k_3$

$\quad -k_1 + 3k_2 = -3k_3$

$\Rightarrow \quad |B| = \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 3+2 = 5$

$|B_1| = \begin{vmatrix} \frac{1}{2}k_3 & 2 \\ -3k_3 & 3 \end{vmatrix} = 7.5\, k_3$

$|B_2| = \begin{vmatrix} 1 & \frac{1}{2}k_3 \\ -1 & -3k_3 \end{vmatrix} = -2.5\, k_3$

$\therefore \quad k_1 = \dfrac{|B_1|}{|B|} = \dfrac{7.5\, k_3}{5} = 1.5\, k_3$

$k_2 = \dfrac{|B_2|}{|B|} = \dfrac{-2.5 k_3}{5} = -0.5\, k_3$

take $k_3 = 2$ then $k_1 = 3$ and $k_2 = -1$.

Thus $3p_1(x) - p_2(x) + 2p_3(x) = 0$ which means that $p_1(x), p_2(x), p_3(x)$ are linearly dependent, i.e. S is linearly dependent

ii- $k_1 p_1(x) + k_2 p(x) = 0$

$\Rightarrow k_1 (1-x) + k_2 (2+3x-2x^2) = 0$

$\Rightarrow k_1 - k_1 x + 2k_2 + 3k_2 x - 2k_2 x^2 = 0$

$\Rightarrow (k_1 + 2k_2) + (-k_1 + 3k_2)x - 2k_2 x^2 = 0$

$\Rightarrow \quad k_1 + 2k_2 = 0 \quad \cdots ④$

$\quad -k_1 + 3k_2 = 0 \quad \cdots ⑤$

$\quad -2k_2 = 0 \quad \cdots ⑥$

multiply equation ⑥ by $-\frac{1}{2}$, we get

$k_2 = 0$ ---(7)

substitute (7) in (4), we get

$k_1 = 0$

Thus $k_1 = 0$ and $k_2 = 0$ which means that $p_1(x)$ and $p_2(x)$ are linearly independent, i.e. T is linearly independent.

<u>Definition</u>: Let V be a vector space over a field $(F, +, \cdot)$ and $S = \{u_1, u_2, \ldots, u_n\}$ is a set of vectors in V, then S is called a basis for V if S is linearly independent and S spans V.

<u>Theorem 4</u>: If $S = \{u_1, u_2, \ldots, u_n\}$ is a basis for a vector space V over a field $(F, +, \cdot)$, then every vector u in V can be ex-pressed in the form $u = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$ in exactly one way.

<u>Proof</u>: Every vector u in V can be written as a linear combinations of the vectors in S since S spans V.

Now assume that there is a vector w in V and w can be expressed as a linear combination in two ways as follows

$w = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$ --- (1) and

$w = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ --- (2)