linear Algebra 1

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Lecture 6

Basis and Dimensional

39. $k_2 = 0$ ---(7)

substitute (7) in (4), we get

$k_1 = 0$

Thus $k_1 = 0$ and $k_2 = 0$ which means that $p_1(x)$ and $p_2(x)$ are linearly independent, i.e. T is linearly independent.

Definition: Let V be a vector space over a field $(F, +, \cdot)$ and $S = \{u_1, u_2, \ldots, u_n\}$ is a set of vectors in V, then S is called a basis for V if S is linearly independent and S spans V.

Theorem 4: If $S = \{u_1, u_2, \ldots, u_n\}$ is a basis for a vector space V over a field $(F, +, \cdot)$, then every vector u in V can be expressed in the form $u = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n$ in exactly one way.

Proof: Every vector u in V can be written as a linear combinations of the vectors in S since S spans V.

Now assume that there is a vector w in V and w can be expressed as a linear combination in two ways as follows

$$w = k_1 u_1 + k_2 u_2 + \cdots + k_n u_n \quad \text{--- (1)} \quad \text{and}$$

$$w = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n \quad \text{--- (2)}$$

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subtracting equation (2) from equation (1), we get

$(k_1-c_1)u_1+(k_2-c_2)u_2+\cdots+(k_n-c_n)u_n=0$

the $k_1-c_1=0, k_2-c_2=0, \ldots, k_n-c_n=0$ since S is linearly independent.

Thus $k_1=c_1, k_2=c_2, \ldots, k_n=c_n$. Hence w can be expressed in exactly one way as a linear combination of the vectors in S.

Example 1: Prove that $S=\{i,j,k\}$ is a basis for the vector space R^3 over the field $(R,+,\cdot)$, where $i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$.

Solution: S is linearly independent since

$c_1 i+c_2 j+c_3 k=(0,0,0)$

$\Rightarrow c_1(1,0,0)+c_2(0,1,0)+c_3(0,0,1)=(0,0,0)$

$\Rightarrow (c_1,0,0)+(0,c_2,0)+(0,0,c_3)=(0,0,0)$

$\Rightarrow (c_1,c_2,c_3)=(0,0,0)$

$\Rightarrow c_1=0, c_2=0$, and $c_3=0$.

Also S spans R^3 since $\forall u=(u_1,u_2,u_3)\in R^3$ we have

$$\begin{aligned} u &= (u_1,u_2,u_3) \\ &= (u_1,0,0)+(0,u_2,0)+(0,0,u_3) \\ &= u_1(1,0,0)+u_2(0,1,0)+u_3(0,0,1) \\ &= u_1 i+u_2 j+u_3 k \end{aligned}$$

Thus S is a basis for the vector space R^3

41. over the field $(R,+,\cdot)$.

<u>Definition</u>: The set $S=\{e_1,e_2,\dots,e_n\}$ where $e_1=(1,0,0,\dots,0)$, $e_2=(0,1,0,\dots,0)$, $\dots$, $e_n=(0,0,\dots,0,1)$ ($e_1,e_2,\dots,e_n$ are n-tuple) is a basis of R^n over the field $(R,+,\cdot)$ which is called the standard basis of R^n over the field $(R,+,\cdot)$.

<u>Example 2</u>: Let $u_1=(1,1,0)$, $u_2=(0,1,1)$, $u_3=(1,0,1)$. Prove that $S=\{u_1,u_2,u_3\}$ is a basis for the vector space R^3 over the field $(R,+,\cdot)$.

<u>Solution</u>:

$k_1u_1+k_2u_2+k_3u_3=(0,0,0)$.

$\Rightarrow k_1(1,1,0)+k_2(0,1,1)+k_3(1,0,1)=(0,0,0)$

$\Rightarrow (k_1,k_1,0)+(0,k_2,k_2)+(k_3,0,k_3)=(0,0,0)$

$\Rightarrow (k_1+k_3,k_1+k_2,k_2+k_3)=(0,0,0)$

$$\Rightarrow \begin{aligned} k_1 \quad\quad + k_3 &= 0 \\ k_1+k_2 \quad\quad &= 0 \\ k_2+k_3 &= 0 \end{aligned}$$

$$\Rightarrow |A|=\begin{vmatrix}1&0&1\\1&1&0\\0&1&1\end{vmatrix}=1\cdot\begin{vmatrix}1&0\\1&1\end{vmatrix}-0\cdot\begin{vmatrix}1&0\\0&1\end{vmatrix}+1\cdot\begin{vmatrix}1&1\\0&1\end{vmatrix}$$

$$=1\cdot(1)-0\cdot(1)+1\cdot(1)$$

$$=1-0+1=2.$$

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$|A_1| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$, $|A_2| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$,

$|A_3| = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$,

$k_1 = \frac{|A_1|}{|A|} = \frac{0}{2} = 0$, $k_2 = \frac{|A_2|}{|A|} = \frac{0}{2} = 0$, $k_3 = \frac{|A_3|}{|A|} = \frac{0}{2} = 0$.

$\Rightarrow$ S is linearly independent.

$\forall\, v = (v_1, v_2, v_3) \in R^3$, we have

$c_1 \cdot u_1 + c_2 \cdot u_2 + c_3 \cdot u_3 = v$

$\Rightarrow c_1 \cdot (1,1,0) + c_2 \cdot (0,1,1) + c_3 \cdot (1,0,1) = (v_1, v_2, v_3)$

$\Rightarrow (c_1, c_1, 0) + (0, c_2, c_2) + (c_3, 0, c_3) = (v_1, v_2, v_3)$

$\Rightarrow (c_1 + c_3, c_1 + c_2, c_2 + c_3) = (v_1, v_2, v_3)$

$$\Rightarrow \begin{aligned} c_1 \quad\;\; + c_3 &= v_1 \\ c_1 + c_2 \quad\;\; &= v_2 \\ c_2 + c_3 &= v_3 \end{aligned}$$

$\Rightarrow |B| = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$,

$|B_1| = \begin{vmatrix} v_1 & 0 & 1 \\ v_2 & 1 & 0 \\ v_3 & 1 & 1 \end{vmatrix} = v_1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0 \cdot \begin{vmatrix} v_2 & 0 \\ v_3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} v_2 & 1 \\ v_3 & 1 \end{vmatrix}$
$= v_1 + v_2 - v_3$,

$|B_2| = \begin{vmatrix} 1 & v_1 & 1 \\ 1 & v_2 & 0 \\ 0 & v_3 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} v_2 & 0 \\ v_3 & 1 \end{vmatrix} - v_1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & v_2 \\ 0 & v_3 \end{vmatrix}$
$= v_2 - v_1 + v_3 = -v_1 + v_2 + v_3$,

$$|B_2| = \begin{vmatrix} 1 & 0 & v_1 \\ 1 & 1 & v_2 \\ 0 & 1 & v_3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & v_2 \\ 1 & v_3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 1 & v_2 \\ 0 & v_3 \end{vmatrix} + v_1 \cdot \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= v_3 - v_2 + v_1 = v_1 - v_2 + v_3$$

$$\therefore\ c_1 = \frac{|B_1|}{|B|} = \frac{v_1 + v_2 - v_3}{2}\ ,\quad c_2 = \frac{|B_2|}{|B|} = \frac{-v_1 + v_2 + v_3}{2}\ ,$$

$$c_3 = \frac{|B_3|}{|B|} = \frac{-v_1 + v_2 + v_3}{2}\ .$$

Thus S spans R^3 over the field $(R, +, \cdot)$.
Therefore S is a basis for R^3 over the field $(R, +, \cdot)$.

<u>Definition</u>: A nonzero vector space V over a field $(R, +, \cdot)$ is called a finite dimensional, if V contains a finite set of vectors $\{v_1, v_2, \ldots, v_n\}$ that forms a basis. If no such finite set exists, then V is called infinite dimensional. The zero vector space is regarded as finite dimensional.

<u>Theorem 5</u>: If V is a finite dimensional vector space and $\{v_1, v_2, \ldots, v_n\}$ is any basis for V, then

i- every set with more than n vectors is linearly dependent.

ii- No set with less than n vectors spans V.

<u>Theorem 6</u>: All bases for a finite dimensional vector space have the same number of vectors.

<u>Definition</u>: The dimension of a finite dimensional vector space V over a field $(F,+,\cdot)$ is denoted by $\dim_F V$ and is defined to be the number of vectors in a basis for V over the field $(F,+,\cdot)$. The zero vector space defined to have a zero dimension.

<u>Examples</u>:

1) $\dim_F F^n = n$, hence we have $\dim_R R^n = n$.

2) $\dim_C C^n = n$ and $\dim_R C^n = 2n$, and

   C^n is of infinite dimension over the field $(Q,+,\cdot)$, for each $n \geq 1$.

3) R^n is of infinite dimension over the field $(Q,+,\cdot)$, for each $n \geq 1$.

4) $\dim_F M_{m\times n}(F) = mn$ where $M_{m\times n}(F)$ is the vector space of all $m\times n$ matrices over $(F,+,\cdot)$, hence we have $\dim_R M_{m\times n}(R) = mn$ and $\dim_Q M_{m\times n}(Q) = mn$.

45. **Theorem 7**: If V is an n-dimensional vector space over a field $(F,+,.)$ and $S=\{u_1,u_2,\dots,u_n\}\subset V$, then S is a basis for V over the field $(F,+,.)$ if either S spans V or S is linearly independent.

§6: Algebra of Subspaces

Definition: Let W_1 and W_2 be subspaces of a vector space V over a field $(F,+,.)$. The set of all vectors u which are in both W_1 and W_2 is called the intersection of the subspaces W_1 and W_2 and is denoted by $W_1 \cap W_2$.

Remark: The intersection $W_1 \cap W_2$ of the subspaces W_1 and W_2 of a vector space V over a field $(F,+,.)$ is never empty since the zero vector 0 belong to both W_1 and W_2.

Theorem 8: The intersection of two subspace W_1 and W_2 of a vector space V over a field $(F,+,.)$ is also a subspace of V over the field $(F,+,.)$.