linear Algebra 1

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Lecture 7

Al gebera of subspace

45\. *Theorem 7:* If V is an n-dimensional vector space over a field $(F,+,\cdot)$ and $S=\{u_1, u_2, \ldots, u_n\} \subset V$, then S is a basis for V over the field $(F,+,\cdot)$ if either S spans V or S is linearly independent.

§6: Algebra of Subspaces

Definition: Let W_1 and W_2 be subspaces of a vector space V over a field $(F,+,\cdot)$. The set of all vectors u which are in both W_1 and W_2 is called the intersection of the subspaces W_1 and W_2 and is denoted by $W_1 \cap W_2$.

Remark: The intersection $W_1 \cap W_2$ of the subspaces W_1 and W_2 of a vector space V over a field $(F,+,\cdot)$ is never empty since the zero vector 0 belong to both W_1 and W_2.

Theorem 8: The intersection of two subspace W_1 and W_2 of a vector space V over a field $(F,+,\cdot)$ is also a subspace of V over the field $(F,+,\cdot)$.

46- **Proof:**

1) The zero vector 0 belongs to W_1 and W_2 which implies that $0 \in W_1 \cap W_2$. Hence $W_1 \cap W_2 \neq \Phi$.

2) $\forall u_1, u_2 \in W_1 \cap W_2$, we have $u_1, u_2 \in W_1$ and $u_1, u_2 \in W_2$ which implies that $u_1 + u_2 \in W_1$ and $u_1 + u_2 \in W_2$ since W_1 and W_2 are subspaces of the vector space V over the field $(F, +, \cdot)$.

Thus $u_1 + u_2 \in W_1 \cap W_2$.

3) $\forall r \in F$ and $\forall v \in W_1 \cap W_2$, we have $r \in F$ and $v \in W_1$; $r \in F$ and $v \in W_2$ which implies that $rv \in W_1$ and $rv \in W_2$ since W_1 and W_2 are subspaces of the vector space V over the field $(F, +, \cdot)$.

Thus by theorem 2, $W_1 \cap W_2$ is a subspace of the vector space V over the field $(F, +, \cdot)$.

Example: Let $W_1 = \{(x, y, z) \in R^3 \mid z = 0\}$ and $W_2 = \{(x, y, z) \in R^3 \mid y = 0\}$ be subspaces of the vector space R^3 over the field $(R, +, \cdot)$. Find the subspace $W_1 \cap W_2$.

47.

Solution:

$(x,y,z) \in W_1 \cap W_2$

$\Rightarrow (x,y,z) \in W_1$ and $(x,y,z) \in W_2$

$\Rightarrow z=0$ and $y=0$

$\Rightarrow W_1 \cap W_2 = \{(x,0,0) \mid x \in R\}$.

Definition: Let W_1 and W_2 be subspaces of a vector space V over a field $(F,+,\cdot)$. The subset W of V consisting of all vectors of the form $ru_1 + su_2$ where $u_1 \in W_1$ and $u_2 \in W_2$ and $r,s \in F$ is called the sum of W_1 and W_2 and denoted by $W_1 + W_2$ (i.e. $W_1 + W_2 = \{ru_1 + su_2 \mid u_1 \in W_1, u_2 \in W_2, r,s \in F\}$).

Theorem 9: The sum of two subspaces W_1 and W_2 of a vector space V over a field $(F,+,\cdot)$ is also a subspace of V over the field $(F,+,\cdot)$.

Proof: 1) The zero vector 0 belongs to W_1 and W_2 which implies that $1\cdot 0 + 1\cdot 0 = 0$ belongs to $W_1 + W_2$ (i.e. $0 \in W_1 + W_2$) i.e. $W_1 + W_2 \neq \Phi$.

2) $\forall (r_1 u_1 + r_2 u_2), (s_1 w_1 + s_2 w_2) \in W_1 + W_2$, (where $u_1, w_1 \in W_1$, $u_2, w_2 \in W_2$, and $r_1, r_2, s_1, s_2 \in F$), we have

48. $(r_1 v_1 + r_2 v_2) + (s_1 w_1 + s_2 w_2)$
$= (r_1 v_1 + s_1 w_1) + (r_2 v_2 + s_2 w_2)$
$= \quad u(r_1 v_1 + s_1 w_1) + u(r_2 v_2 + s_2 w_2) \in W_1 + W_2$,
since $u \in F$ and $(r_1 v_1 + s_1 w_1) \in W_1$ and $(r_2 v_2 + s_2 w_2) \in W_2$ because W_1 and W_2 are subspaces of the vector space V over the field $(F,+,\cdot)$.

3.) $\forall\, r \in F$ and $\forall\, (r_1 v_1 + r_2 v_2) \in W_1 + W_2$, we have $r\cdot(r_1 v_1 + r_2 v_2) = r r_1 v_1 + r r_2 v_2 \in W_1 + W_2$ since $r r_1, r r_2 \in F$, $v_1 \in W_1$, $v_2 \in W_2$.

Thus by theorem 2, $W_1 + W_2$ is a subspace of the vector space V over the field $(F,+,\cdot)$.

Example: Let $W_1 = \{(x,0,0) \mid x \in R\}$ and $W_2 = \{(0,y,0) \mid y \in R\}$ be subspaces of the vector space R^3 over the field $(R,+,\cdot)$. Find the subspace $W_1 + W_2$.

Solution:

$$W_1 + W_2 = \{r v_1 + s v_2 \mid r, s \in R \text{ and } v_1 \in W_1, v_2 \in W_2\}$$
$$= \{r(x,0,0) + s(0,y,0) \mid r,s,x,y \in R\}$$
$$= \{(rx,0,0) + (0,sy,0) \mid r,s,x,y \in R\}$$
$$= \{(rx, sy, 0) \mid r,s,x,y \in R\}$$
$$= \{(x_1, x_2, 0) \mid x_1, x_2 \in R\}$$