linear Algebra 1

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Lecture 8

Orthonormal Basis

49. § 7: Orthonormal Bases in R^n

Definition: A set $T = \{v_1, v_2, \dots, v_m\}$ in R^n is called orthogonal if $v_i \cdot v_j = 0$ for $i \neq j$ and $i, j = 1, 2, \dots, m$.

Definition: A set $T = \{v_1, v_2, \dots, v_m\}$ in R^n is called orthonormal if $v_i \cdot v_j = 0$ for $i \neq j$ and $i, j = 1, 2, \dots, m$; and $\|v_j\| = 1$ $\forall j = 1, \dots, m$.

Theorem 10: Let $T = \{v_1, v_2, \dots, v_m\}$ be an orthogonal set of nonzero vectors in R^n. Then T is linearly independent.

Corollary: Let $T = \{v_1, v_2, \dots, v_m\}$ be an orthonormal set of vectors in R^n is linearly independent.

Example: Let $v_1 = (0.6, 0.8, 0, 0)$, $v_2 = (-0.8, 0.6, 0, 0)$, $v_3 = (0, 0, 1, 0)$. Prove that the set $T = \{v_1, v_2, v_3\}$ is orthonormal.

Solution:

$$v_1 \cdot v_2 = (0.6, 0.8, 0, 0) \cdot (-0.8, 0.6, 0, 0)$$
$$= 0.6 \times (-0.8) + 0.8 \times 0.6 + 0 \times 0 + 0 \times 0$$

$= -0.48 + 0.48 + 0 + 0 = 0$.

50. $v_1 \cdot v_3 = (0.6, 0.8, 0, 0) \cdot (0, 0, 1, 0)$

$= 0.6 \times 0 + 0.8 \times 0 + 0 \times 1 + 0 \times 0 = 0 + 0 + 0 + 0 = 0$.

$v_2 \cdot v_3 = (-0.8, 0.6, 0, 0) \cdot (0, 0, 1, 0)$

$= -0.8 \times 0 + 0.6 \times 0 + 0 \times 1 + 0 \times 0 = 0 + 0 + 0 + 0 = 0$.

$\therefore$ The set $T = \{v_1, v_2, v_3\}$ is an orthogonal set.

$\| v_1 \| = \sqrt{(0.6)^2 + (0.8)^2 + (0)^2 + (0)^2}$

$= \sqrt{0.36 + 0.64 + 0 + 0} = \sqrt{1} = 1$

$\| v_2 \| = \sqrt{(-0.8)^2 + (0.6)^2 + (0)^2 + (0)^2}$

$= \sqrt{0.64 + 0.36 + 0 + 0} = \sqrt{1} = 1$

$\| v_3 \| = \sqrt{(0)^2 + (0)^2 + (1)^2 + (0)^2}$

$= \sqrt{0 + 0 + 1 + 0} = \sqrt{1} = 1$

$\therefore$ The set $T = \{v_1, v_2, v_3\}$ is an orthonormal set.

Definition: A basis $B = \{v_1, v_2, \dots, v_n\}$ for R^n is called an orthogonal basis for R^n if B is an orthogonal set.

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Example: Let $v_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $v_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $v_3 = (0, 0, 1)$. Prove that the basis $B = \{v_1, v_2, v_3\}$ is an orthonormal basis.

Solution:

$$u_1 \cdot u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$
$$= -\frac{1}{2} + \frac{1}{2} + 0 = 0 .$$
$$u_1 \cdot u_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1) = \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1)$$
$$= 0 + 0 + 0 = 0 .$$
$$u_2 \cdot u_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1)$$
$$= -\frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1) = 0 + 0 + 0 = 0$$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthogonal basis.

$$\| u_1 \| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\| u_2 \| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\| u_3 \| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{0 + 0 + 1} = \sqrt{1} = 1 .$$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthonormal basis.

§ 8 : Linear Transformation and Matrices

Definition: Let V and W be two vector spaces over the same field $(F, +, \cdot)$.
A function $f : V \longrightarrow W$ is called a linear