linear Algebra 1

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Lecture 9

Linear transformation

basis.

Solution:

$$u_1 \cdot u_2 = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \cdot \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right)$$
$$= -\tfrac{1}{2} + \tfrac{1}{2} + 0 = 0.$$

$$u_1 \cdot u_3 = \left(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1) = \tfrac{1}{\sqrt{2}}(0) + \tfrac{1}{\sqrt{2}}(0) + 0(1)$$
$$= 0 + 0 + 0 = 0.$$

$$u_2 \cdot u_3 = \left(-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1)$$
$$= -\tfrac{1}{\sqrt{2}}(0) + \tfrac{1}{\sqrt{2}}(0) + 0(1) = 0 + 0 + 0 = 0$$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthogonal basis.

$$\|u_1\| = \sqrt{\left(\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\|u_2\| = \sqrt{\left(-\tfrac{1}{\sqrt{2}}\right)^2 + \left(\tfrac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\tfrac{1}{2} + \tfrac{1}{2} + 0} = \sqrt{1} = 1.$$

$$\|u_3\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{0 + 0 + 1} = \sqrt{1} = 1.$$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthonormal basis.

§8: Linear Transformation and Matrices

Definition: Let V and W be two vector spaces over the same field $(F, +, \cdot)$.
A function $f: V \longrightarrow W$ is called a linear

52. transformation (or linear mapping or vector space homomorphism) if f satisfies the following:

1) $f(w+u) = f(w) + f(u) \quad \forall\, w, u \in V$

2) $f(ru) = r f(u) \quad \forall\, r \in F$ and $\forall\, u \in V$.

شرطا التحويل الخطي [illegible]

Definition: Let V be a vector space over a field $(F, +, \cdot)$. A linear transformation $f: V \longrightarrow V$ is called a linear operator $\boxed{\text{on}}$ V (or linear transformation $\boxed{\text{on}}$ V).

Example: Show that the function $f: R^2 \longrightarrow R^2$ defined by $f(x, y) = (x - y, 3x + y) \;\forall (x, y) \in R^2$, is a linear transformation on the vector space R^2 over the field $(R, +, \cdot)$.

on بمعنى من R^2 لنفسه

Solution: $\forall\, (x_1, y_1), (x_2, y_2) \in R^2$, we have

$$f((x_1, y_1) + (x_2, y_2))$$
$$= f(x_1 + x_2, y_1 + y_2)$$
$$= ((x_1 + x_2) - (y_1 + y_2), 3(x_1 + x_2) + (y_1 + y_2))$$
$$= (x_1 + x_2 - y_1 - y_2, 3x_1 + 3x_2 + y_1 + y_2)$$
$$= (x_1 - y_1 + x_2 - y_2, 3x_1 + y_1 + 3x_2 + y_2)$$
$$= (x_1 - y_1, 3x_1 + y_1) + (x_2 - y_2, 3x_2 + y_2)$$
$$= f(x_1, y_1) + f(x_2, y_2),$$ and

$\forall\, r \in R$ and $\forall (x, y) \in R^2$, we have

$f(r(x,y))$
$= f(rx, ry)$
$= (rx - ry, 3(rx) + ry)$
$= (r(x-y), r(3x+y))$
$= r(x-y, 3x+y)$
$= r\, f(x,y)$

$\therefore$ f is a linear transformation on the vector space R^2 over the field $(R, +, \cdot)$.

Theorem 11: If V and W be two vector spaces over a field $(F, +, \cdot)$ and $f: V \longrightarrow W$ is a linear transformation from V to W, then

i) $f(0_V) = 0_W$

ii) $f(-v) = -f(v) \quad \forall v \in V$.

Definition: If V and W be two vector spaces over a field $(F, +, \cdot)$ and $f: V \longrightarrow W$ is a linear transformation from V to W, then the kernel of f, which is denoted by $\ker f$, is the set of all $v \in V$ satisfying that $f(v) = 0_W$, i.e.
$\ker f = \{ v \in V \mid f(v) = 0_W \}$.

Definition: If V and W are two vector spaces over a field $(F,+,\cdot)$ and $f: V \to W$ is a linear transformation from V to W, then the range of f is the set of all $f(v)$, $v \in V$ and will be denoted by Range f, i.e. Range $f = \{ f(v) \mid v \in V \}$.

Theorem 12: If V and W are two vector spaces over a field $(F,+,\cdot)$ and $f: V \to W$ is a linear transformation from V to W, then

i) The kernel of f is a subspace of V.

ii) The range of f is a subspace of W.

Example: Let $f: R^3 \longrightarrow R^3$ defined by $f(x,y,z) = (x+y, 2z, z)$ $\forall (x,y,z) \in R^3$ be a linear transformation on the vector space R^3 over the field $(R,+,\cdot)$. Find ker f and Range f.

Solution:

$(x,y,z) \in \ker f$

$\Rightarrow f(x,y,z) = (0,0,0)$

$\Rightarrow (x+y, 2z, z) = (0,0,0)$

$\Rightarrow x+y=0$ and $2z=0$

$\Rightarrow y=-x$ and $z=0$

Thus $\ker f = \{ (x,-x,0) \mid x \in R \}$.

Range $f = \{ f(x,y,z) \mid (x,y,z) \in R^3 \}$

55. $= \{(x+y, 2z, z) \mid x, y, z \in R\}$

$= \{(u, 2z, z) \mid u, z \in R\}$.

Definition: Let f be a linear transformation on a vector space V over a field $(F,+,\cdot)$ and suppose $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V, and let f defined on the elements of B as follows:

$$f(v_1) = k_{11} v_1 + k_{12} v_2 + \dots + k_{1n} v_n$$
$$f(v_2) = k_{21} v_1 + k_{22} v_2 + \dots + k_{2n} v_n$$
$$\vdots$$
$$f(v_n) = k_{n1} v_1 + k_{n2} v_2 + \dots + k_{nn} v_n$$

Then the matrix

$$\begin{pmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{pmatrix}$$

which is denoted by $m_B(f)$ or $[f]_B$ is called the matrix representation of f relative to the basis B.

Remark: Let B be a basis of a vector space V over a field $(F,+,\cdot)$. Then any vector $v \in V$ can be written uniquely in the form $v = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$ where $B = \{v_1, v_2, \dots, v_n\}$