linear Algebra 1

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Lecture 10

Matrix Representation

basis.

Solution:

$u_1 \cdot u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$\quad = -\frac{1}{2} + \frac{1}{2} + 0 = 0.$

$u_1 \cdot u_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1) = \frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1)$

$\quad = 0 + 0 + 0 = 0.$

$u_2 \cdot u_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right) \cdot (0, 0, 1)$

$\quad = -\frac{1}{\sqrt{2}}(0) + \frac{1}{\sqrt{2}}(0) + 0(1) = 0 + 0 + 0 = 0$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthogonal basis.

$\| u_1 \| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$

$\| u_2 \| = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2} = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = \sqrt{1} = 1.$

$\| u_3 \| = \sqrt{(0)^2 + (0)^2 + (1)^2} = \sqrt{0 + 0 + 1} = \sqrt{1} = 1.$

$\therefore$ The basis $B = \{u_1, u_2, u_3\}$ is an orthonormal basis.

§ 8: Linear Transformation and Matrices

Definition: Let V and W be two vector spaces over the same field $(F, +, \cdot)$. A function $f : V \longrightarrow W$ is called a linear

52. transformation (or linear mapping or vector space homomorphism) if f satisfies the following:

1) $f(w+u) = f(w) + f(u) \quad \forall w, u \in V$
2) $f(ru) = r f(u) \quad \forall r \in F$ and $\forall u \in V$.

شرطي التحويل الخطي

Definition: Let V be a vector space over a field $(F, +, \cdot)$. A linear transformation $f : V \longrightarrow V$ is called a linear operator $\boxed{\text{on}}$ V (or linear transformation $\boxed{\text{on}}$ V).

Example: Show that the function $f : R^2 \longrightarrow R^2$ defined by $f(x, y) = (x - y, 3x + y) \ \forall (x, y) \in R^2$ is a linear transformation on the vector space R^2 over the field $(R, +, \cdot)$.

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Solution: $\forall (x_1, y_1), (x_2, y_2) \in R^2$, we have

$$f((x_1, y_1) + (x_2, y_2))$$
$$= f(x_1 + x_2, y_1 + y_2)$$
$$= ((x_1 + x_2) - (y_1 + y_2), 3(x_1 + x_2) + (y_1 + y_2))$$
$$= (x_1 + x_2 - y_1 - y_2, 3x_1 + 3x_2 + y_1 + y_2)$$
$$= (x_1 - y_1 + x_2 - y_2, 3x_1 + y_1 + 3x_2 + y_2)$$
$$= (x_1 - y_1, 3x_1 + y_1) + (x_2 - y_2, 3x_2 + y_2)$$
$$= f(x_1, y_1) + f(x_2, y_2), \text{ and}$$

$\forall r \in R$ and $\forall (x, y) \in R^2$, we have

$= \{(x+y, 2z, z) \mid x, y, z \in R\}$

$= \{(u, 2z, z) \mid u, z \in R\}$.

Definition : Let f be a linear transformation on a vector space V over a field $(F, +, \cdot)$ and suppose $B = \{v_1, v_2, \ldots, v_n\}$ is a basis of V, and let f defined on the elements of B as follows:

$$f(v_1) = k_{11} v_1 + k_{12} v_2 + \cdots + k_{1n} v_n$$
$$f(v_2) = k_{21} v_1 + k_{22} v_2 + \cdots + k_{2n} v_n$$
$$\vdots$$
$$f(v_n) = k_{n1} v_1 + k_{n2} v_2 + \cdots + k_{nn} v_n$$

Then the matrix

$$\begin{pmatrix} k_{11} & k_{21} & \cdots & k_{n1} \\ k_{12} & k_{22} & \cdots & k_{n2} \\ \vdots & \vdots & & \vdots \\ k_{1n} & k_{2n} & \cdots & k_{nn} \end{pmatrix}$$

which is denoted by $m_B(f)$ or $[f]_B$ is called the matrix representation of f relative to the basis B.

Remark: Let B be a basis of a vector space V over a field $(F, +, \cdot)$. Then any vector $v \in V$ can be written uniquely in the form $v = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n$ where $B = \{v_1, v_2, \ldots, v_n\}$

56. and the coordinate vector of v relative to the basis B is denoted and defined by

$$[v]_B = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = (k_1, k_2, \dots, k_n)^T$$

<u>Example 1</u>: Let $g : R^2 \to R^2$ defined by $g(x,y) = (x-y, 3x+y)$ $\forall (x,y) \in R^2$ be a linear operator on the vector space R^2 over the field $(R,+,\cdot)$. Find the representation matrices $m_B(g)$ and $m_S(g)$ of the linear operator g relative to each of the bases $B = \{u_1, u_2\}$ and $S = \{e_1, e_2\}$ for R^2, where $u_1 = (1,1)$, $u_2 = (-1,0)$, $e_1 = (1,0)$, $e_2 = (0,1)$. مطلوب ثانيٍ الحل

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<u>Solution</u>:

$\forall (a,b) \in R^2$, we have

$$\begin{pmatrix} a \\ b \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 \\ k_1 \end{pmatrix} + \begin{pmatrix} -k_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} k_1 - k_2 \\ k_1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} k_1 - k_2 &= a \\ k_1 &= b \end{aligned}$$

Thus $k_1 = b$ and $k_2 = b - a$.

Hence $(a,b) = b u_1 + (b-a) u_2$

Therefore $[(a,b)]_B = \begin{pmatrix} b \\ b-a \end{pmatrix}$

$g(u_1) = g(1,1) = (1-1, 3+1) = (0,4)$
$= 4u_1 + (4-0)u_2 = 4u_1 + 4u_2$ and
$g(u_2) = g(-1,0) = (-1-0, -3+0) = (-1,-3)$
$= -3u_1 + (-3-(-1))u_2 = -3u_1 - 2u_2$.

Therefore $[g]_B = m_B(g) = \begin{pmatrix} 4 & -3 \\ 4 & -2 \end{pmatrix}$.

Now $\forall (a,b) \in R^2$, we have
$g(e_1) = g(1,0) = (1-0, 3+0) = (1,3) = 1 \cdot e_1 + 3e_2$
and $g(e_2) = g(0,1) = (0-1, 0+1) = (-1,1) = -e_1 + e_2$.
Therefore $[g]_S = m_S(g) = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$.

Example 2: Let V be a vector space over a field $(F,+,\cdot)$, and let $B = \{2, t, e^{2t}, te^{2t}\}$ be a basis of V and let d be the differential operator on V, i.e. $d(v) = \frac{dv}{dt}$. Find the representation matrix of d relative to the basis B. نتمنى لك مستقبلا لتحقق فيه مطالب

Solution:

$d(2) = 0 = 0 \cdot 2 + 0 \cdot t + 0 \cdot e^{2t} + 0 \cdot te^{2t}$
$d(t) = 1 = \frac{1}{2} \cdot 2 + 0 \cdot t + 0 \cdot e^{2t} + 0 \cdot te^{2t}$
$d(e^{2t}) = 2e^{2t} = 0 \cdot 2 + 0 \cdot t + 2 \cdot e^{2t} + 0 \cdot te^{2t}$
$d(te^{2t}) = e^{2t} + 2te^{2t} = 0 \cdot 2 + 0 \cdot t + 1 \cdot e^{2t} + 2 \cdot te^{2t}$

اذا كانت الناس

58. Thus $[d]_B = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Theorem 13: Let V be a vector space over a field $(F, +, \cdot)$ and let B be a basis for V and let f_1 and f_2 be any linear operators on V and $k \in F$, then we have

i) $m_B(f_1 + f_2) = m_B(f_1) + m_B(f_2)$ that is $[f_1 + f_2]_B = [f_1]_B + [f_2]_B$.

ii) $m_B(kf_1) = k\, m_B(f_1)$ that is $[kf_1]_B = k[f_1]_B$

iii) $m_B(f_1 \circ f_2) = m_B(f_1) \cdot m_B(f_2)$ that is $[f_1 \circ f_2]_B = [f_1]_B \cdot [f_2]_B$.

Example: Let V be a vector space over a field $(F, +, \cdot)$ and let $B = \{e^{5t}, te^{5t}\}$ and let d and h be two linear operators on V defined by $d(v) = 3\frac{d}{dt} v \quad \forall v \in V$ and $h(e^{5t}) = 3e^{5t} + te^{5t}$ and $h(te^{5t}) = e^{5t} - te^{5t}$. Find $m_B(d + 3h)$ and $m_B(d \circ h)$.

Solution:

$d(e^{5t}) = 3\frac{d}{dt} e^{5t} = 3(5e^{5t}) = 15e^{5t}$
$= 15e^{5t} + 0 \cdot te^{5t}$.

$d(te^{5t}) = 3\frac{d}{dt}(te^{5t}) = 3(e^{5t} + 5te^{5t})$
$= 3e^{5t} + 15te^{5t}$.

59. Thus $m_B(d) = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}$.

$h(e^{5t}) = 3e^{5t} + te^{5t}$
$h(te^{5t}) = e^{5t} - te^{5t}$

Thus $m_B(h) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$.

By theorem 13, $m_B(d+3h) = m_B(d) + m_B(3h)$

$= m_B(d) + 3\,m_B(h) = \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix} + 3\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix} + \begin{pmatrix} 9 & 3 \\ 3 & -3 \end{pmatrix} = \begin{pmatrix} 24 & 6 \\ 3 & 12 \end{pmatrix}$, and

$m_B(d \circ h) = m_B(d) \cdot m_B(h)$

$= \begin{pmatrix} 15 & 3 \\ 0 & 15 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 48 & 12 \\ 15 & -15 \end{pmatrix}$