linear Algebra 2

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Lecture1

Eigenvectors and Eigenvalues

جبر خطي
مرحلة ٢/ علوم رياضيات

Linear Algebra with Applications I

References

1) Introductary Linear Algebra with Applications, by Bernard Kolman.
2) Elementary Linear Algebra, by Howard Anton.
3) Linear Algebra, by Fraleigh
4) Theory and Problems of Linear Algebra, by Seymour Lipschutz.
5) 3000 Solved Problems in Linear Algebra, by Seymour Lipschutz.
6) Introduction to Linear Algebra, by Franz Hohn.

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§1 : Characteristic equation, Eigenvalues, Eigenvectors

Definition: The characteristic equation of $n \times n$ matrix A is the nth-degree polynomial equation $\det\boxed{(A-\lambda I) = 0}$.
The nth-degree polynomial $\det(A-\lambda I)$ is called the characteristic polynomial of the matrix A.

Definition: Let A be an $n \times n$ matrix over a field K. A scalar $\lambda \in K$ is called an eigenvalue of A if there exists a nonzero column vector $v \in K^n$ (i.e. v is an $n \times 1$ matrix) for which $\boxed{A \cdot v = \lambda \cdot v}$.

Definition: Let A be an $n \times n$ matrix over a field K. A column vector $v \in K^n$ is called an eigenvector of A corresponding to an eigenvalue λ of A if $A \cdot v = \lambda \cdot v$.
The set of all such vectors is a subspace of K^n over K called the eigenspace of λ.

2. Remarks:

1) The terms characteristic value and proper value are some times used instead of eigenvalue.

2) The terms characteristic vector and proper vector are some times used instead of eigenvector.

3) If v is an eigenvector of an $n \times n$ matrix A corresponding to an eigenvalue $\lambda \in K$ of A, then $\forall k \in K$ we have $k.v$ is also an eigenvector of A corresponding to λ since $A.k.v = k.A.v = k.(\lambda.v) = \lambda.(k.v)$.

Example: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ be a 2×2 matrix over R. Find the $\boxed{\text{eigenvalues}}$ of A and then find the $\boxed{\text{eigenspace}}$ of each eigenvalue.

Solution: To find a scalar $\lambda \in R$ and a nonzero column vector $v = \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ such that $A.v = \lambda.v$, let $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda.\begin{pmatrix} x \\ y \end{pmatrix}$, then

$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda.\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, thus

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

3. $\Rightarrow \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \ldots (1)$

Therefore (1) has a nonzero solution (for x and y) iff $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$\Longleftrightarrow (1-\lambda)(2-\lambda) - 6 = 2 - 3\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1) = 0$

$\Longleftrightarrow (\lambda - 4) = 0$ or $(\lambda + 1) = 0$

$\Longleftrightarrow \lambda = 4$ or $\lambda = -1$

Thus $\lambda = 4$ and $\lambda = -1$ are the eigenvalues of A.

(1) (2)

Let $\lambda = 4$ in (1), we get

$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} -3x + 2y = 0 \\ 3x - 2y = 0 \end{matrix}$

$\Rightarrow 2y = 3x \Rightarrow y = \frac{3}{2} x$. Thus the vector

اذا أخذ $x = 2$, $y = 3$

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda_1 = 4$ of A. Therefore the eigenspace of $\lambda_1 = 4$ is

$V_1 = \left\{ \begin{pmatrix} 2k \\ 3k \end{pmatrix} \middle| \, k \in R \right\}$.

Let $\lambda = -1$ in (1), we get

$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2x + 2y = 0 \\ 3x + 3y = 0 \end{matrix}$

$\Rightarrow x + y = 0 \Rightarrow y = -x$. Thus the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. is an eigenvector of A corresponding to the eigenvalue $\lambda_2 = -1$ of A. Therefore the eigenspace of $\lambda_2 = -1$ is

$$V_2 = \left\{ \begin{pmatrix} k \\ -k \end{pmatrix} \,\middle|\, k \in R \right\}.$$

<u>Theorem 1</u>: Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$ (i.e. the eigenvalues of an $n \times n$ matrix are the values of λ which satisfy the characteristic equation of the matrix A).

<u>Example 1</u>: Find the characteristic polynomial, the characteristic equation, and the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix}$.

<u>Solution</u>:

$$A - \lambda I = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 1 & 6 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 2-\lambda & 5 \\ 1 & 6-\lambda \end{pmatrix}.$$

Thus $\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 5 \\ 1 & 6-\lambda \end{vmatrix}$

$= (2-\lambda)(6-\lambda) - 5 = 12 - 8\lambda + \lambda^2 - 5$

$= \lambda^2 - 8\lambda + 7$

Therefore the characteristic polynomial of the matrix A is $\lambda^2 - 8\lambda + 7$ and the characteristic equation is $\lambda^2 - 8\lambda + 7 = 0$.

$2+6$ $\quad 2\times 6 - 5 = 7$

5- $\lambda^2 - 8\lambda + 7 = 0 \Rightarrow (\lambda - 7)(\lambda - 1) = 0$

$\Rightarrow \lambda - 7 = 0$ or $\lambda - 1 = 0$

$\Rightarrow \lambda = 7$ or $\lambda = 1$.

Thus $\lambda = 7$ and $\lambda = 1$ are the eigenvalues of A.

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Example 2: Find the characteristic polynomial and the characteristic equation and the eigenvalues of the matrix $D = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}$

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Solution:

$D - \lambda I = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{pmatrix}$.

Thus $\det(D - \lambda I) = \begin{vmatrix} 4-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda)$

$= \lambda^2 - 5\lambda + 4$.

Therefore the characteristic polynomial of the matrix D is $\lambda^2 - 5\lambda + 4$ and the characteristic equation of the matrix D is $\lambda^2 - 5\lambda + 4 = 0$.

$\lambda^2 - 5\lambda + 4 = 0 \Rightarrow (\lambda - 4)(\lambda - 1) = 0$

$\Rightarrow \lambda - 4 = 0$ or $\lambda - 1 = 0$

$\Rightarrow \lambda = 4$ or $\lambda = 1$

Thus $\lambda = 4$ and $\lambda = 1$ are the eigenvalues of D.

6. <u>Example 3</u>: Find the characteristic polynomial, the characteristic equation, and the eigenvalues of the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 3 & 0 \\ 9 & 12 & 4 \end{pmatrix}_{3\times 3}$$

<u>Solution</u>:

$$E - \lambda I = \begin{pmatrix} 1 & 0 & 0 \\ 5 & 3 & 0 \\ 9 & 12 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 5 & 3 & 0 \\ 9 & 12 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 0 & 0 \\ 5 & 3-\lambda & 0 \\ 9 & 12 & 4-\lambda \end{pmatrix}$$

Thus $\det(E - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 5 & 3-\lambda & 0 \\ 9 & 12 & 4-\lambda \end{vmatrix}$

$$= (1-\lambda)\begin{vmatrix} 3-\lambda & 0 \\ 12 & 4-\lambda \end{vmatrix} - 0 + 0$$

$$= (1-\lambda)(3-\lambda)(4-\lambda)$$

$$= (1-\lambda)(12 - 7\lambda + \lambda^2)$$

هلدالاله بالحد $12+4+3$

$$= 12 - 7\lambda + \lambda^2 - 12\lambda + 7\lambda^2 - \lambda^3$$

بموجب [illegible] لازم تغ [illegible] $1+3+4=8$ يعني det

$$= -\lambda^3 + 8\lambda^2 - 19\lambda + 12 \qquad - + - + = \text{ ونقرب بسالب}$$

[illegible]

لازم تقرب بسالب
وبعد موجب
وبعد سالب
حسب المعادلة

Therefore the characteristic polynomial of the matrix E is $-\lambda^3 + 8\lambda^2 - 19\lambda + 12$ and the characteristic equation of the matrix E is $-\lambda^3 + 8\lambda^2 - 19\lambda + 12 = 0$

$$\cancel{-\lambda^3 + 8\lambda^2 + 19\lambda + 12} = 0 \Rightarrow (1-\lambda)(3-\lambda)(4-\lambda) = 0.$$

$$(-\lambda+1)(\cancel{-\lambda^2 + 7\lambda - 12})$$

$\Rightarrow 1-\lambda=0$ or $3-\lambda=0$ or $4-\lambda=0$

$\Rightarrow \lambda=1$ or $\lambda=3$ or $\lambda=4$.

Thus $\lambda=1$ and $\lambda=3$ and $\lambda=4$ are the eigenvalues of the matrix E.

Example 4: Find the characteristic polynomial, the characteristic equation, and the eigenvalues of the matrix

$$B=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}.$$

Solution:

$$B-\lambda I=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & -1-\lambda\end{pmatrix}$$

Thus $\det(B-\lambda I)=\begin{vmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & -1-\lambda\end{vmatrix}$

$$=(1-\lambda)\begin{vmatrix}2-\lambda & -1\\ 1 & -1-\lambda\end{vmatrix}+1\begin{vmatrix}3 & -1\\ 2 & -1-\lambda\end{vmatrix}+4\begin{vmatrix}3 & 2-\lambda\\ 2 & 1\end{vmatrix}$$

$$=(1-\lambda)((2-\lambda)(-1-\lambda)+1)+(3(-1-\lambda)+2)$$
$$+4(3-2(2-\lambda))$$

8. $= (1-\lambda)(-2-\lambda+\lambda^2+1) + (-3-3\lambda+2) + (12-16+8\lambda)$

$= -2-\lambda+\lambda^2+1+2\lambda+\lambda^2-\lambda^3-\lambda-1-3\lambda-4+8\lambda$

$= -\lambda^3 + 2\lambda^2 + 5\lambda - 6$

Therefore the characteristic polynomial of the matrix B is $-\lambda^3+2\lambda^2+5\lambda-6$ and the characteristic equation of the matrix B is $-\lambda^3+2\lambda^2+5\lambda-6=0$.

$-\lambda^3+2\lambda^2+5\lambda-6=0 \quad (-\lambda+1)(\quad\quad) \qquad -\lambda^2-\lambda+6$

$\Rightarrow (-\lambda+1)(-\lambda-2)(-\lambda+3)=0$

$\Rightarrow -\lambda+1=0$ or $-\lambda-2=0$ or $-\lambda+3=0$

$\Rightarrow \lambda=1$ or $\lambda=-2$ or $\lambda=3$

Thus $\lambda=1$ and $\lambda=-2$ and $\lambda=3$ are the eigenvalues of the matrix B.

Remarks :

1) The characteristic equation of a 2×2 matrix D is $a_2\lambda^2+a_1\lambda+a_0=0$ where $a_2=1$, $a_1=-d_{11}-d_{22}$, and $a_0=|D|$.

2) The characteristic equation of a 3×3 matrix E is $a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0$ where $a_3=-1$, $a_2=e_{11}+e_{22}+e_{33}$,

$$a_1 = -\begin{vmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{vmatrix} - \begin{vmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{vmatrix} - \begin{vmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{vmatrix}, \text{ and}$$

$a_0 = |E|$.