linear Algebra 2

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Lecture4

Digonalization matrices

14\. **Properties of Similarity**

1) Any matrix is similar to itself (i.e. any matrix A is similar to the matrix A).
2) If a matrix A is similar to a matrix B, then the matrix B is similar to the matrix A.
3) If a matrix A is similar to a matrix B and the matrix B is similar to a matrix C, then the matrix A is similar to the matrix C.

§3 : Diagonalization

Definition: An $n\times n$ matrix A is called diagonalizable if A is similar to a diagonal matrix B.

Theorem 2: If A is an $n\times n$ matrix, then A is diagonalizable iff A has n linealy independent eigenvectors.

Example: Show that the matrix $A=\begin{pmatrix}12 & 10\\ 1 & 3\end{pmatrix}$ is diagonalizable.

Solution: $Au=\lambda u \Rightarrow Au=\lambda I u \Rightarrow (A-\lambda I)u=0$

$$\Rightarrow \left(\begin{pmatrix}12 & 10\\ 1 & 3\end{pmatrix}-\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right)\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$

$\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

15- $\Rightarrow \begin{pmatrix} 12-\lambda & 10 \\ 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ --- (1)

Equation (1) has non trivial solution iff

$\begin{vmatrix} 12-\lambda & 10 \\ 1 & 3-\lambda \end{vmatrix} = (12-\lambda)(3-\lambda) - 10$

$= 36 - 12\lambda - 3\lambda + \lambda^2 - 10 = \lambda^2 - 15\lambda + 26$

$= (\lambda - 13)(\lambda - 2) = 0$

$\Leftrightarrow \lambda - 13 = 0$ or $\lambda - 2 = 0$

$\Leftrightarrow \lambda = 13$ or $\lambda = 2$

Thus the eigenvalues of the matrix A are $\lambda_1 = 13$ and $\lambda_2 = 2$.

Let $\lambda = 13$ in (1), we get

$\begin{pmatrix} -1 & 10 \\ 1 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -x + 10y \\ x - 10y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} -x + 10y = 0 \\ x - 10y = 0 \end{matrix} \Rightarrow x = 10y$.

Thus the vector $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix A corresponding (مطابق) to the eigenvalue $\lambda = 13$.

Let $\lambda = 2$ in (1), we get

$\begin{pmatrix} 10 & 10 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 10x + 10y \\ x + y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} 10x + 10y = 0 \\ x + y = 0 \end{matrix} \Rightarrow y = -x$.

Thus the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of ~~the matrix A corresponding~~ to the eigenvalue $\lambda = 2$.

16. Let $P = \begin{pmatrix} 10 & 1 \\ 1 & -1 \end{pmatrix}$. Then the diagonal matrix B similar to A is

$$B = P^{-1}AP = \begin{pmatrix} 10 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 12 & 10 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{-11} & \frac{-1}{-11} \\ \frac{-1}{-11} & \frac{10}{-11} \end{pmatrix} \begin{pmatrix} 12 & 10 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{11} & \frac{1}{11} \\ \frac{1}{11} & \frac{-10}{11} \end{pmatrix} \begin{pmatrix} 130 & 2 \\ 13 & -2 \end{pmatrix} = \begin{pmatrix} \frac{143}{11} & 0 \\ 0 & \frac{22}{11} \end{pmatrix} = \begin{pmatrix} 13 & 0 \\ 0 & 2 \end{pmatrix}$$

Example: Show that the matrix $A = \begin{pmatrix} 10 & 8 \\ 6 & 2 \end{pmatrix}$ is diagonalizable.

Solution: $Av = \lambda v \Rightarrow Av = \lambda I v \Rightarrow (A - \lambda I)v = 0$

$$\Rightarrow \left(\begin{pmatrix} 10 & 8 \\ 6 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 10-\lambda & 8 \\ 6 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \cdots \ (1)$$

Equation (1) has non trivial solution iff

$$\begin{vmatrix} 10-\lambda & 8 \\ 6 & 2-\lambda \end{vmatrix} = (10-\lambda)(2-\lambda) - 48$$

$$= 20 - 10\lambda - 2\lambda + \lambda^2 - 48$$

$$= \lambda^2 - 12\lambda - 28$$

$$= (\lambda - 14)(\lambda + 2) = 0$$

$$\Longleftrightarrow \quad \lambda - 14 = 0 \quad \text{or} \quad \lambda + 2 = 0$$

$$\Longleftrightarrow \quad \lambda = 14 \quad \text{or} \quad \lambda = -2$$

Thus the eigenvalues of the matrix A are $\lambda_1 = 14$ and $\lambda_2 = -2$.

Let $\lambda = 14$ in (1), we get

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17. $\begin{pmatrix} -4 & 8 \\ 6 & -12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -4x + 8y \\ 6x - 12y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{matrix} -4x + 8y = 0 \\ 6x - 12y = 0 \end{matrix} \Rightarrow x = 2y .$$

Thus the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda = 14$.

Let $\lambda = -2$ in (1), we get

$$\begin{pmatrix} 12 & 8 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 12x + 8y \\ 6x + 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} 12x + 8y = 0 \\ 6x + 4y = 0 \end{matrix} \Rightarrow y = -\frac{3}{2}x .$$

Thus the vector $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda = -2$

Let $P = \begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}$. Then the diagonal matrix

$-6-2=-8$

B similar to A is

$$B = P^{-1} A P = \begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 10 & 8 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-3}{-8} & \frac{-2}{-8} \\ \frac{-1}{-8} & \frac{2}{-8} \end{pmatrix} \begin{pmatrix} 10 & 8 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 1 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{2}{8} \\ \frac{1}{8} & \frac{-2}{8} \end{pmatrix}\begin{pmatrix} 28 & -4 \\ 14 & 6 \end{pmatrix} = \begin{pmatrix} \frac{112}{8} & 0 \\ 0 & \frac{-16}{8} \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 0 \\ 0 & -2 \end{pmatrix} .$$

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18. **Example:** Show that whether the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ is diagonalizable or not. هل هذا المصفوفة قابلة للاقطار،

Solution:

$AV = \lambda V \Rightarrow AV - \lambda V = 0 \Rightarrow AV - \lambda I V = 0$

$\Rightarrow (A - \lambda I)V = 0$

$\Rightarrow \left(\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdots (1)$

Equation (1) has non trivial solution iff

$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) = 0$

$\Rightarrow 2-\lambda = 0$ or $2-\lambda = 0$

$\Rightarrow \lambda = 2$ or $\lambda = 2$.

Thus the eigenvalues of the matrix A are $\lambda_1 = 2$ and $\lambda_2 = 2$

Let $\lambda = 2$ in equation (1), we get

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} y = 0 \longrightarrow \\ 0 = 0 \end{matrix}$ يعني لازم صفر

Thus the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and each vector $\begin{pmatrix} x \\ 0 \end{pmatrix}$ where x is nonzero real number are eigenvectors corresponding to the eigenvalue $\lambda = 2$ of the matrix A.

Since A does not have two linearly independent eigenvectors then by theorem 2, A is not

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والحدود
والكلية

19. diagonalizable.

بس اذا مصفوفات تبين هذه الصفات يجب ان تكون Similar

<u>Theorem 3</u>: All the roots of the characteristic polynomial of a symmetric matrix are real numbers.

يعني الارقام التي على القطر الاول
تساوي العدد المقابل

<u>Corollary</u>: If A is a symmetric matrix and all the eigenvalues of A are distinct, then A is diagonalizable.

<u>Example</u>: Show that whether the matrix $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ is diagonalizable or not.

<u>Solution:</u>

$AU = \lambda U \Rightarrow AU - \lambda U = 0 \Rightarrow AU - \lambda I U = 0$

$\Rightarrow (A - \lambda I)U = 0$

$\Rightarrow \left(\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \cdots (1)$

Equation (1) has nontrivial solution iff

$\begin{vmatrix} 5-\lambda & 1 \\ 1 & 5-\lambda \end{vmatrix} = (5-\lambda)(5-\lambda) - 1 = 25 - 5\lambda - 5\lambda + \lambda^2 - 1$

$= \lambda^2 - 10\lambda + 24 = 0$

$\Rightarrow (\lambda - 6)(\lambda - 4) = 0$

$\Rightarrow \lambda - 6 = 0$ or $\lambda - 4 = 0 \Rightarrow \lambda = 6$ or $\lambda = 4$.

Thus the eigenvalues of the matrix A are

$\lambda_1 = 6$ and $\lambda_2 = 4$.

Let $\lambda = 6$ in equation (1), we get.

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20. $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} -x+y=0 \\ x-y=0 \end{matrix} \Rightarrow y = x$

Thus the vector $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 6$ of the matrix A.

Let $\lambda = 4$ in equation (1), we get

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x+y=0 \\ x+y=0 \end{matrix} \Rightarrow y = -x$

Thus, the vector $w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue $\lambda = 4$ of the matrix A.

Let $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, then $P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

Thus the diagonal matrix B similar to A is

$B = P^{-1}AP = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 6 & -4 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 4 \end{pmatrix}$.