

LINEAR ALGEBRA - LECTURE 1

Matrices and Matrix Operations

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Stage	Fourth Stage

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1. Matrix notation and dimensions

A matrix is a rectangular arrangement of scalars. If A has m rows and n columns, then A is an $m \times n$ matrix and its entry in row i and column j is denoted a_{ij} . The ordered pair (m,n) is the size or order of the matrix. Matrix notation is especially useful because large systems of equations, data tables, images, graphs, and linear maps can all be represented compactly.

Special matrices

A row matrix has one row, a column matrix has one column, and a square matrix has the same number of rows and columns. The zero matrix has all entries equal to zero. The identity matrix I_n has ones on the main diagonal and zeros elsewhere. A diagonal matrix has zeros off the main diagonal. A scalar matrix is a diagonal matrix whose diagonal entries are equal. A triangular matrix has zeros entirely above or below the main diagonal.

Equality, addition, and scalar multiplication

Two matrices A and B are equal when they have the same size and $a_{ij}=b_{ij}$ for every i,j. Addition is defined only for equal-sized matrices: $(A+B)_{ij}=a_{ij}+b_{ij}$. Scalar multiplication is defined by $(cA)_{ij}=ca_{ij}$. Matrix addition is commutative and associative, and scalar multiplication distributes over matrix addition.

Example 1

Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 & 5 \\ 1 & 0 & -3 \end{bmatrix}$. Then $A+B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 3 & 1 \end{bmatrix}$ and $2A = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 6 & 8 \end{bmatrix}$. Both results remain 2×3 matrices.

Computer-science viewpoint

A grayscale image can be stored as a matrix whose entries represent pixel intensities. A dataset can be stored as a matrix in which rows represent observations and columns represent features. Adjacency matrices represent graphs, while transformation matrices are used in computer graphics.

2. Matrix multiplication

Let A be $m \times n$ and B be $n \times p$. Their product AB is defined and has size $m \times p$. The (i,j) entry is obtained by multiplying corresponding entries of row i of A and column j of B and adding the products. Thus $(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. The inner dimensions must agree.

Study guidance

Read each definition together with its conditions. In linear algebra, many errors arise not from arithmetic but from applying an operation when dimensions or hypotheses do not permit it. Write dimensions beside matrices during calculations and keep notation consistent.

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Example 2

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$,
 $AB = \begin{bmatrix} 1(2)+2(1) & 1(0)+2(5) \\ 3(2)+4(1) & 3(0)+4(5) \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 20 \end{bmatrix}$.

Properties of multiplication

Matrix multiplication is associative: $A(BC) = (AB)C$. It is distributive over addition: $A(B+C) = AB+AC$ and $(A+B)C = AC+BC$. In general it is not commutative: AB need not equal BA . Also $AB=0$ does not necessarily imply $A=0$ or $B=0$.

Identity and powers

For every compatible matrix A , $I_m A = A$ and $A I_n = A$. For a square matrix, powers are defined by $A^2 = AA$, $A^3 = AAA$, and $A^0 = I$. Powers occur naturally in repeated state transitions and recurrence models.

Example 3: noncommutativity

Take $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, while $BA = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Hence AB is not equal to BA .

Practical note

When multiplying several matrices, check dimensions before doing arithmetic. A dimension chain such as $(3 \times 4)(4 \times 2)(2 \times 5)$ is valid and ultimately produces a 3×5 matrix.

3. Transpose and symmetry

The transpose A^T is obtained by converting rows into columns. If A is $m \times n$, then A^T is $n \times m$. Important identities are $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$, and $(AB)^T = B^T A^T$. A square matrix is symmetric when $A^T = A$ and skew-symmetric when $A^T = -A$.

Inverse of a matrix

A square matrix A is invertible if a matrix A^{-1} exists such that $AA^{-1} = A^{-1}A = I$. The inverse, when it exists, is unique. A matrix with no inverse is singular. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad-bc$ is nonzero.

Example 4

For $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\det(A) = 2(1) - 1(1) = 1$, so $A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Direct multiplication verifies $AA^{-1} = I$.

Inverse laws

If A and B are invertible, then $(AB)^{-1} = B^{-1} A^{-1}$. Also $(A^T)^{-1} = (A^{-1})^T$. These order-reversal rules parallel the transpose rule for products.

Solving with an inverse

If A is invertible, $Ax = b$ implies $x = A^{-1}b$. Although this formula is conceptually important, numerical algorithms generally solve systems using elimination or matrix factorizations rather than explicitly computing A^{-1} .

Verification strategy

A completed calculation should be checked independently whenever possible. Matrix products can be checked by dimensions and selected entries; inverses by multiplication with the original matrix; solutions by substitution; and basis or rank conclusions by row reduction.

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Connection to algorithms

Matrix inversion and multiplication appear in graphics, estimation, optimization, control, and machine learning. Efficient implementations exploit structure such as sparsity, symmetry, triangular form, or block structure.

4. Elementary matrices and review

An elementary matrix is obtained by performing one elementary row operation on an identity matrix. Left multiplication by an elementary matrix performs the same row operation on another matrix. A square matrix is invertible exactly when it can be reduced to I by elementary row operations.

Finding an inverse by Gauss-Jordan

To compute A^{-1} , form $[A|I]$ and row-reduce the left block to I . If the reduction succeeds, the right block becomes A^{-1} . If a zero pivot structure prevents the left side from becoming I , the matrix is singular.

Example 5

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, begin $[1 \ 2 \ | \ 1 \ 0; \ 3 \ 5 \ | \ 0 \ 1]$. Replace R_2 by $R_2 - 3R_1$ to get $[1 \ 2 \ | \ 1 \ 0; \ 0 \ -1 \ | \ -3 \ 1]$. Scale R_2 by -1 , then eliminate the 2 above the second pivot. The result is $[1 \ | \ -5 \ 2; \ 0 \ -1]$, hence $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

Summary

Matrices organize linear information. Valid operations depend on dimensions. Matrix multiplication uses row-column products and is generally noncommutative. Transpose changes orientation; inverses undo nonsingular square transformations; elementary row operations provide an algorithmic route to inverses.

Exercises

1) Compute $3A - 2B$ for two 2×3 matrices of your choice. 2) Multiply a 2×3 matrix by a 3×2 matrix. 3) Verify $(AB)^T = B^T A^T$ numerically. 4) Find the inverse of $\begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$. 5) Use Gauss-Jordan to invert $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. 6) Give a computer-science example where a matrix represents data.

Revision points

Before moving to the next lecture, be able to state the main definitions without notes, solve at least one representative example from beginning to end, explain the meaning of the answer, and identify one application in computing or data analysis.