

LINEAR ALGEBRA - LECTURE 2

Determinants and Their Properties

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Stage	Fourth Stage

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1. Determinant concept

The determinant assigns a scalar to every square matrix. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad - bc$. A determinant is not defined for a nonsquare matrix. Algebraically, it detects singularity. Geometrically, its absolute value gives the factor by which a linear transformation scales area or volume, while the sign indicates whether orientation is preserved or reversed.

Minors and cofactors

For an $n \times n$ matrix, the minor M_{ij} is the determinant obtained after deleting row i and column j . The cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$. Expansion along row i is $\det(A) = \sum_j a_{ij} C_{ij}$; a corresponding formula holds for columns.

Example 1

For $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \\ 2 & 0 & 1 \end{bmatrix}$, expansion along the first row gives $\det(A) = 1(1 \cdot 1 - 4 \cdot 0) - 2(3 \cdot 1 - 4 \cdot 2) + 0 = 1 - 2(-5) = 11$.

Choosing an expansion

Cofactor expansion is easiest along a row or column containing many zeros. For large dense matrices, direct cofactor expansion is inefficient; elimination-based methods are preferable.

Geometric meaning

If a 2×2 matrix maps the unit square to a parallelogram, $|\det(A)|$ equals the parallelogram's area. In three dimensions, $|\det(A)|$ gives the volume scale factor.

2. Determinant properties

$\det(I) = 1$ and $\det(A^T) = \det(A)$. Interchanging two rows changes the sign of the determinant. Multiplying a row by k multiplies the determinant by k . Adding a multiple of one row to another does not change the determinant. These rules make row reduction a powerful evaluation method.

Triangular matrices

For an upper or lower triangular matrix, the determinant is the product of the diagonal entries. After reducing a matrix to triangular form while tracking row swaps and row scalings, its determinant is easy to compute.

Study guidance

Read each definition together with its conditions. In linear algebra, many errors arise not from arithmetic but from applying an operation when dimensions or hypotheses do not permit it. Write dimensions beside matrices during calculations and keep notation consistent.

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Product rule

For square matrices of equal order, $\det(AB) = \det(A)\det(B)$. Consequently, if A is invertible then $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$, so $\det(A^{-1}) = 1/\det(A)$.

Example 2

Suppose row reduction changes A into U using two row swaps and no row scaling. Then $\det(A) = \det(U)$, because two sign reversals cancel. If U has diagonal 2, -3, 5, then $\det(A) = -30$.

Singularity test

A square matrix is invertible if and only if $\det(A)$ is nonzero. Therefore $\det(A) = 0$ is equivalent to singularity, linear dependence of the columns, and the existence of a nonzero solution of $Ax = 0$.

3. Determinants and linear systems

For $Ax = b$ with square A , a nonzero determinant guarantees a unique solution for every b . This is one form of the Invertible Matrix Theorem, which links determinants, pivots, rank, independence, and invertibility.

Cramer's rule

If $\det(A)$ is nonzero, then $x_i = \det(A_i)/\det(A)$, where A_i is obtained by replacing column i of A with b . Cramer's rule gives a closed formula but is computationally expensive for large systems.

Example 3

Solve $2x + y = 5$ and $x - y = 1$. $\det(A) = \det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -3$. For x , $\det\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix} = -6$, so $x = 2$. For y , $\det\begin{pmatrix} 2 & 5 \\ 1 & 1 \end{pmatrix} = -3$, so $y = 1$.

Adjugate formula

For an invertible square matrix, $A^{-1} = (1/\det(A)) \text{adj}(A)$, where $\text{adj}(A)$ is the transpose of the cofactor matrix. This formula is important theoretically, though elimination is more practical computationally.

Numerical caution

A very small determinant does not by itself provide a complete measure of numerical conditioning, especially because determinant magnitude depends on scaling and dimension. Numerical stability is better assessed using norms, condition numbers, and stable factorization methods.

Verification strategy

A completed calculation should be checked independently whenever possible. Matrix products can be checked by dimensions and selected entries; inverses by multiplication with the original matrix; solutions by substitution; and basis or rank conclusions by row reduction.

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4. Applications and review

Determinants appear in coordinate transformations, Jacobians, geometry, eigenvalue problems, orientation tests, and volume computations. In computational geometry, determinant signs can help determine which side of a directed line a point lies on.

Characteristic determinants

Eigenvalues of A are roots of $\det(A - \lambda I) = 0$. Thus determinants provide the bridge from matrix algebra to spectral analysis.

Summary

The determinant is a scalar invariant of a square matrix. Cofactor expansion defines it recursively, while row operations provide efficient calculation. A nonzero determinant is equivalent to invertibility, and determinant identities explain how linear transformations scale oriented volume.

Exercises

1) Compute $\det\begin{pmatrix} 1 & 4 & 7 \\ 2 & 6 & 1 \end{pmatrix}$. 2) Evaluate a 3×3 determinant by cofactor expansion. 3) Recompute it using elimination. 4) Explain the determinant change caused by each elementary row operation. 5) Use Cramer's rule on a 2×2 system. 6) Prove from $\det(AB) = \det(A)\det(B)$ that $\det(A^{-1}) = 1/\det(A)$.

Concept check

If a matrix has two equal rows, its determinant is zero: swapping those rows leaves the matrix unchanged but should reverse the determinant's sign, forcing $\det(A) = -\det(A)$. This illustrates how structural properties can yield determinant conclusions without computation.

Revision points

Before moving to the next lecture, be able to state the main definitions without notes, solve at least one representative example from beginning to end, explain the meaning of the answer, and identify one application in computing or data analysis.