

LINEAR ALGEBRA - LECTURE 3

Systems of Linear Equations

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Stage	Fourth Stage

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1. Linear equations and systems

A linear equation in variables $x_1, \dots, x_n$ has the form $a_1x_1 + \dots + a_nx_n = b$. A system is a collection of such equations. In matrix notation it is $Ax=b$, where A is the coefficient matrix, x is the unknown vector, and b is the constant vector.

Augmented matrix

The augmented matrix $[A|b]$ contains all information needed for solving the system. Elementary row operations preserve the solution set: row interchange, multiplication of a row by a nonzero scalar, and replacement of a row by itself plus a multiple of another row.

Geometric interpretation

In two variables, each nondegenerate linear equation represents a line. Two equations may intersect at one point, coincide, or be parallel. These correspond respectively to a unique solution, infinitely many solutions, or no solution.

Example 1

For $x+y=3$ and $2x-y=0$, the augmented matrix is $[1 \ 1 \ | \ 3; \ 2 \ -1 \ | \ 0]$. Replacing R_2 by R_2-2R_1 gives $[1 \ 1 \ | \ 3; \ 0 \ -3 \ | \ -6]$, hence $y=2$ and $x=1$.

Why elimination works

Row operations create equivalent equations. The goal is to transform the system into a form in which variables can be determined directly, while preserving exactly the same solution set.

2. Echelon forms

A matrix is in row echelon form when all nonzero rows precede zero rows, each leading entry lies to the right of the leading entry above it, and entries below each leading entry are zero. In reduced row echelon form, every pivot equals 1 and is the only nonzero entry in its column.

Gaussian elimination

Gaussian elimination produces echelon form and then uses back substitution. Gauss-Jordan elimination continues until reduced row echelon form is obtained. RREF is unique, although the sequence of row operations used to reach it is not.

Study guidance

Read each definition together with its conditions. In linear algebra, many errors arise not from arithmetic but from applying an operation when dimensions or hypotheses do not permit it. Write dimensions beside matrices during calculations and keep notation consistent.

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Example 2

Solve $x+y+z=6$, $2x-y+z=3$, $x+2y-z=2$. Eliminate x from rows 2 and 3, then eliminate y from row 3. Back substitution gives $z=3$, $y=2$, $x=1$.

Pivot and free variables

A pivot variable corresponds to a pivot column. A nonpivot variable is free and may be assigned a parameter. If a consistent system contains free variables, it has infinitely many solutions.

Inconsistency

A row of the form $[0 \ 0 \ \dots \ 0 \ | \ c]$ with c nonzero represents the impossible equation $0=c$. Its appearance proves that the system has no solution.

3. Homogeneous systems

A homogeneous system has the form $Ax=0$. It is always consistent because $x=0$ is a solution. A nonzero or nontrivial solution exists exactly when there is at least one free variable.

Parametric vector form

When free variables occur, the solution set can be written as a linear combination of parameter vectors. For example, $x+2y-z=0$ gives $x=-2y+z$, so $(x,y,z)=y(-2,1,0)+z(1,0,1)$.

Rank

The rank of a matrix is the number of pivot positions. It is also the dimension of the row space and column space. For $Ax=b$, consistency requires $\text{rank}(A)=\text{rank}([A|b])$.

Solution classification

For n unknowns: if $\text{rank}(A)<\text{rank}([A|b])$, there is no solution. If ranks are equal to n , the solution is unique. If ranks are equal but less than n , there are infinitely many solutions.

Verification strategy

A completed calculation should be checked independently whenever possible. Matrix products can be checked by dimensions and selected entries; inverses by multiplication with the original matrix; solutions by substitution; and basis or rank conclusions by row reduction.

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Example 3

A system with 5 unknowns and rank 3 has 2 free variables if it is consistent. Its homogeneous solution set therefore has two independent parameters.

4. Computational issues

In numerical computation, Gaussian elimination is implemented with pivoting. Partial pivoting exchanges rows to use a large available pivot, reducing the effects of round-off and avoiding division by very small numbers.

LU factorization

Elimination can be organized as $A=LU$, where L is lower triangular and U is upper triangular. Once this factorization is available, multiple systems $Ax=b$ with the same A but different b vectors can be solved efficiently.

Applications

Linear systems appear in circuit analysis, interpolation, least squares, graphics, network flow models, finite-difference methods, optimization, and machine learning. Large systems are often sparse, so specialized algorithms store and process only nonzero entries.

Summary

Row operations transform systems without changing their solutions. Pivots determine rank and basic variables; free variables control degrees of freedom. Rank comparison classifies consistency, while numerical implementations add pivoting for stability.

Exercises

1) Solve a 3×3 system by Gaussian elimination. 2) Find the RREF of $[[1,2,3],[2,4,7]]$. 3) Construct an inconsistent system. 4) Express a homogeneous solution in parametric vector form. 5) For 7 unknowns and rank 4, determine the number of free variables. 6) Explain why pivoting is useful.

Revision points

Before moving to the next lecture, be able to state the main definitions without notes, solve at least one representative example from beginning to end, explain the meaning of the answer, and identify one application in computing or data analysis.