

LINEAR ALGEBRA - LECTURE 4

Vector Spaces and Subspaces

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Stage	Fourth Stage

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1. Vector-space structure

A vector space V over a field F is a set equipped with vector addition and scalar multiplication satisfying closure, associativity, commutativity of addition, additive identity, additive inverses, distributive laws, compatibility of scalar multiplication, and scalar identity. These axioms abstract the behavior of ordinary coordinate vectors.

Examples

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$. The set $M_{m,n}$ of all $m \times n$ real matrices is a vector space. The set P_n of real polynomials of degree at most n is a vector space. Continuous real-valued functions on an interval also form a vector space.

Nonexample

The set $\{(x,y): x \geq 0\}$ is not a vector space because multiplying $(1,0)$ by -1 produces $(-1,0)$, which is not in the set. Failure of a single axiom is enough to reject a proposed vector space.

Zero and negatives

The zero vector is unique. Every vector has a unique additive inverse. Also $0v=0$, $c0=0$, and $(-1)v=-v$; these familiar facts follow from the axioms rather than being separate assumptions.

Computer-science relevance

Vectors can represent features, signals, states, coefficients, or embeddings. The vector-space viewpoint allows the same algebraic reasoning to apply to many different data representations.

2. Subspaces

A subset W of V is a subspace if W itself is a vector space under the operations inherited from V . The standard subspace test requires W to be nonempty and closed under linear combinations: $au+bv$ must belong to W for all u,v in W and scalars a,b .

Study guidance

Read each definition together with its conditions. In linear algebra, many errors arise not from arithmetic but from applying an operation when dimensions or hypotheses do not permit it. Write dimensions beside matrices during calculations and keep notation consistent.

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Example 1

$W = \{(x, y, z) : x + y + z = 0\}$ is a subspace of $\mathbb{R}^3$. If u and v satisfy the equation, then the coordinate sum of $au + bv$ is $a(0) + b(0) = 0$.

Example 2: not a subspace

$S = \{(x, y) : x + y = 1\}$ is not a subspace because $(0, 0)$ is not in S . Geometrically, it is a line not passing through the origin.

Intersection and sum

The intersection of any collection of subspaces is a subspace. For subspaces U and W , $U + W = \{u + w : u \in U, w \in W\}$ is also a subspace. If $U \cap W = \{0\}$, the sum is direct and may be written $U \oplus W$.

Solution spaces

The null space $N(A) = \{x : Ax = 0\}$ is a subspace of $\mathbb{R}^n$. This is a fundamental reason homogeneous systems are central in linear algebra.

3. Span and independence

The span of vectors $v_1, \dots, v_k$ is the set of all linear combinations $c_1v_1 + \dots + c_kv_k$. It is the smallest subspace containing those vectors. A set spans V when every vector in V can be written as such a combination.

Testing membership in a span

To test whether b belongs to $\text{span}\{v_1, \dots, v_k\}$, solve $[v_1 \dots v_k]c = b$. Consistency means b is in the span.

Linear independence

Vectors $v_1, \dots, v_k$ are linearly independent if $c_1v_1 + \dots + c_kv_k = 0$ has only the trivial coefficient solution. Otherwise the vectors are dependent. A set containing the zero vector is always dependent.

Example 3

Vectors $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 2)$ are dependent because the third is the sum of the first two.

Verification strategy

A completed calculation should be checked independently whenever possible. Matrix products can be checked by dimensions and selected entries; inverses by multiplication with the original matrix; solutions by substitution; and basis or rank conclusions by row reduction.

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Pivot-column criterion

The pivot columns of a matrix form a linearly independent subset of the original columns. They also provide a basis for the column space when selected from the original matrix rather than from its RREF.

4. Basis, coordinates, and dimension

A basis is a linearly independent spanning set. Every vector in a finite-dimensional vector space has a unique coordinate representation relative to a chosen basis. The number of basis vectors is the dimension.

Example 4

For $W = \{(x, y, z) : x + y + z = 0\}$, write $z = -x - y$. Then $(x, y, z) = x(1, 0, -1) + y(0, 1, -1)$. These two vectors are independent, so they form a basis and $\dim(W) = 2$.

Dimension facts

Any independent set in an n -dimensional space contains at most n vectors. Any spanning set contains at least n vectors. A set of exactly n vectors in an n -dimensional space is a basis if it is either known to be independent or known to span.

Coordinates

If $B = \{b_1, \dots, b_n\}$ is a basis and $v = c_1 b_1 + \dots + c_n b_n$, then $[v]_B = (c_1, \dots, c_n)^T$ is the coordinate vector of v relative to B . Changing basis changes coordinates but not the underlying vector.

Exercises

1) Test whether $\{(x, y, z) : x - 2y + z = 0\}$ is a subspace. 2) Find a basis for its solution space. 3) Test whether three given vectors are independent. 4) Find coordinates relative to a nonstandard basis of $\mathbb{R}^2$. 5) State $\dim(P_4)$ and $\dim(M_{2 \times 3})$. 6) Explain how redundancy in a dataset relates to linear dependence.

Revision points

Before moving to the next lecture, be able to state the main definitions without notes, solve at least one representative example from beginning to end, explain the meaning of the answer, and identify one application in computing or data analysis.