

1.2 Geometric (Graphical) Solution

Feasible Solution: It's the point which is satisfying the constraints of LPP. The set of all such point is the feasible region it should be noted that not every LPP has feasible solution (region).

Optimal Solution: It's the feasible solution which makes the objective function as large (small) as possible.

Example (1.5): Find the set of points that maximize

$$Z = 8x + 10y$$

s.t.

$$2x + 3y \leq 6$$

$$x + 2y \geq 6$$

$$x, y \geq 0$$

Solution: Let us graph the constraints of the LPP, by make the inequality to equality.

$$L1: 2x + 3y = 6 \rightarrow (3,0), (0,2).$$

$$L2: x + 2y = 6 \rightarrow (6,0), (0,3).$$

See figure (1.1).

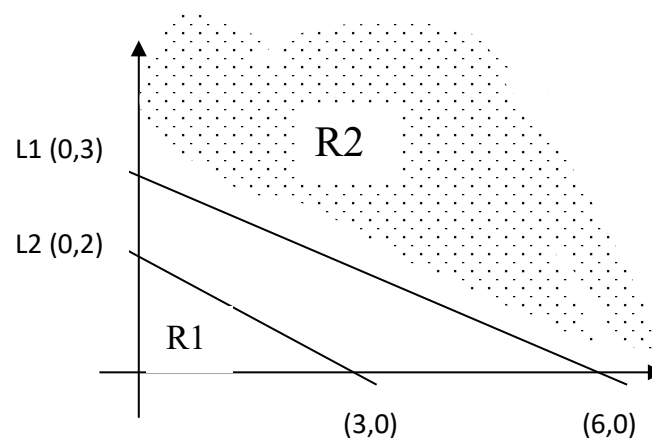


Figure (1.1)

Its clear that there are no point satisfying all inequality, so this LPP has no feasible solution.

Example (1.6): Find the set of points that maximize

$$Z = 8x_1 + 10x_2$$

s.t.

$$2x_1 + x_2 \leq 50$$

$$x_1 + 2x_2 \leq 70,$$

$$x_1, x_2 \geq 0$$

Let us graph the constraint,

$$L1: 2x_1 + x_2 = 50 \rightarrow (25,0), (0,50).$$

$$L2: x_1 + 2x_2 = 70 \rightarrow (70,0), (0,35).$$

$$2x_1 + x_2 = 50$$

$$\pm 2x_1 \pm 4x_2 = \pm 140$$

$$-3x_2 = -90 \rightarrow x_2 = 30, x_1 = 10$$

$\therefore$ intersection point C(10,30).

the feasible region is O(0,0), (0,35), C(10, 30) and E (25,0).

See figure (1.2).

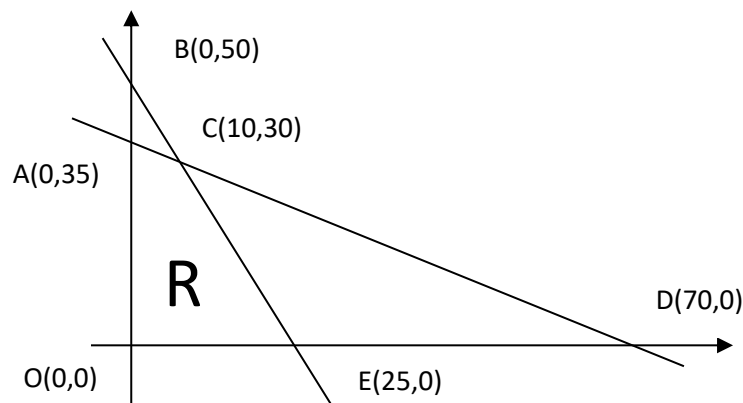


Figure (1.2)

Note: since there are infinitely many feasible solutions in the region, we cannot examine the value of objective function at every point, we shall be able to find optimal solution without examining all feasible solutions.

Line Segment: It's the line joining the distinct points x_1 and $x_2 \in \mathbb{R}^n$ is the set of all points of the form $\lambda x_1 + (1-\lambda) x_2$, $0 \leq \lambda \leq 1$.

Convex Set: it's the set $S \neq \emptyset$, $S \subseteq \mathbb{R}^n$ s.t. if the line segment joining any two points in S also lies entirely in S , as in figure (1.3).

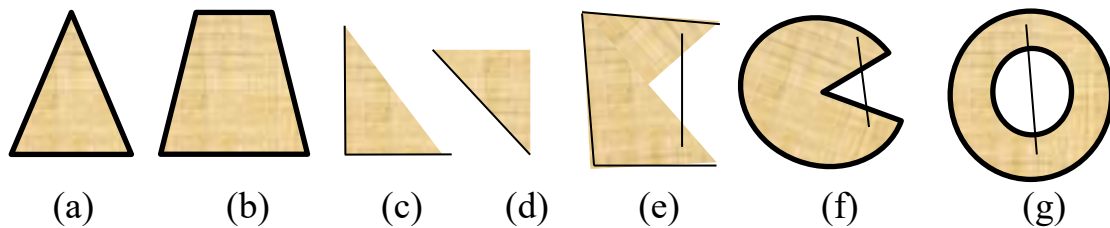


Figure (1.3)

(a), (b), (c) and (d) are convex sets.

(e), (f), (q) are not convex sets.

Convex sets are either bounded or unbounded. A bounded Convex set is one can be enclosed by a sufficiently large rectangle but unbounded set is one that cannot be so enclosed. The convex figure (1-a), (1-b) are bounded, but in (1-c) and (1-d) are unbounded.

Extreme Point: It's a point $\in S$ which is not interior point of any line segment in S .

In Figure (1-d) the convex set has no extreme point.

Theorem (1.1): Let S be the feasible region of a LPP.

- (a). If S is bounded, then the objective function $Z = ax + by$ assumes both a maximum and minimum value on S those values occur at extreme points of S .
- (b). If S is unbounded, then there is may or may not be maximum or minimum values on S . If maximum or minimum value does exist on S it occurs at an extreme point.

The procedure for solving a LPP in two variables geometrically is as follow:

Step1: Sketch the feasible region S.

Step2: Determine all the extreme points of S.

Step3: Evaluate the objective function at each extreme point.

Step4: Choose an extreme point at which the objective function is largest (smallest) for maximum (minimum) problem.

Example (1.7): Consider Example (6) again. S is bounded and its extreme points are O(0,0), A(0,35), C(10,30), E(25,0). Tablue the extreme point as in table (1.3):

Table (1.3)

Points	$Z = 8x_1 + 10x_2$
O(0,0)	0
A(0,30)	350
C(10,30)	380
E(25,0)	200

Notice that the value of Z is maximum at point C(10,30), thus the optimal solution is $x_1 = 10$ and $x_2 = 30$ for $Z = 380$.

Example (1.8): Find the optimal solution of the following LPP.

s.t.

$$\text{Min. } Z = 20x_1 + 30x_2$$

$$x_1 + x_2 \geq 50$$

$$x_1 \leq 30$$

$$x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

Solution:

$$L_1: x_1, x_2 = 50 \rightarrow (50, 0), (0, 50)$$

$$L_2: x_1 = 30$$

$$L_3: x_2 = 40$$

$$\text{From } L_1 \text{ and } L_3 \rightarrow A(10, 40)$$

$$L_2 \text{ and } L_3 \rightarrow B(30, 40)$$

$$L_1 \text{ and } L_2 \rightarrow C(30, 20)$$

Figure (1.4) shows the graphical method.

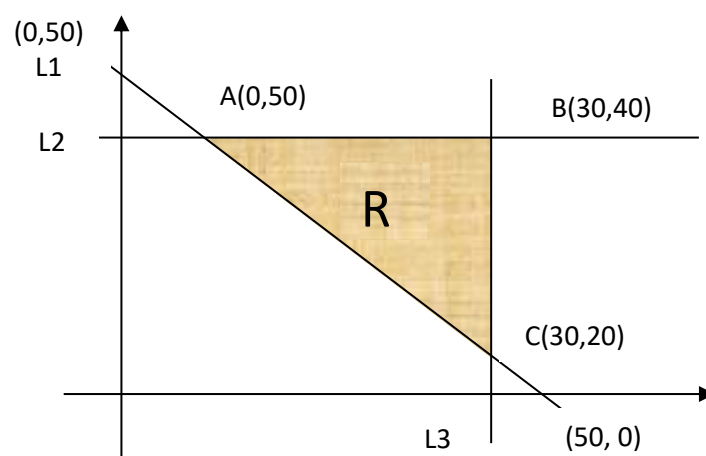


Figure (1.4)

Table (1.4) shows the results of points testing:

Table (1.4)

Points	Z
A(10,40)	1400
B(30,40)	1800
C(30,20)	1200

The value of $Z=1200$ is minimum at the point C(30,20)

In general, a LPP may have:

1. No feasible solution, that is there are no points satisfying all constraints.
2. A unique optimal solution.
3. More than one optimal solution.
4. No maximum (or minimum) value in the feasible region, that is, if may be possible to choose a point in the feasible region to make the objective function as large (or as small) as we please.

Example (1.9): Solve the following LPP geometrically.

$$\text{Min. } Z = 30x_1 + 40x_2$$

s.t.

$$2x_1 + x_2 \geq 12$$

$$x_1 + x_2 \geq 9$$

$$x_1 + 3x_2 \geq 15$$

$$x_1, x_2 \geq 0$$

Solution:

$$L_1: 2x_1 + x_2 = 12 \rightarrow (6, 0), (0, 12).$$

$$L_2: x_1 + x_2 = 9 \rightarrow (9, 0), (0, 9).$$

$$L_3: x_1 + 3x_2 = 15 \rightarrow (15, 0), (0, 5).$$

From L_1 and $L_2 \rightarrow A(3, 6)$

L_2 and $L_3 \rightarrow B(6, 3)$

Figure (1.5) shows the graphical method.

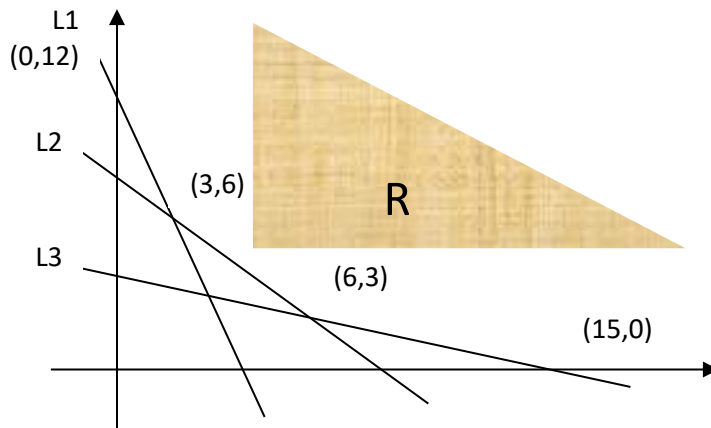


Figure (1.5)

Table (1.5) shows the results of points testing:

Table (1.5)

Points	$Z = 30x_1 + 40x_2$
(0,12)	$0 + 480 = 480$
(3,6)	$90 + 240 = 330$
(6,3)	$180 + 120 = 300$
(15,0)	$450 + 0 = 450$

Example (1.10): Solve the following LPP geometrically.

$$\text{Min. } Z = 20000x_1 + 25x_2$$

s.t.

$$2x_1 + 3x_2 \leq 12$$

$$40x_1 + 60x_2 \geq 300$$

$$x_1, x_2 \geq 0$$

Solution:

$$L_1: 2x_1 + 3x_2 = 12 \rightarrow (6,0), (0, 4).$$

$$L_2: 40x_1 + 60x_2 = 300$$

$$4x_1 + 6x_2 = 30 \rightarrow (7.5, 0), (0, 5).$$

Figure (1.5) shows the graphical method.

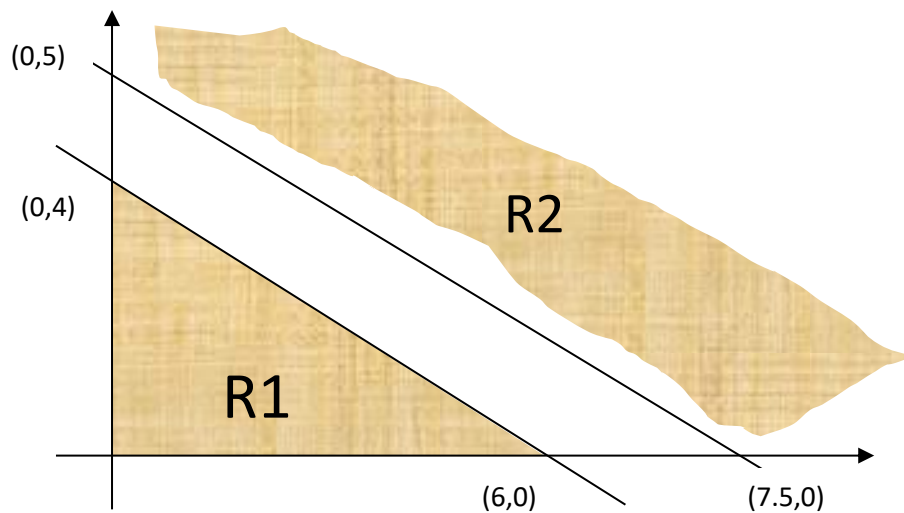


Figure (1.5)

This LPP has no feasible region, so has no feasible solution.

Note: As we notice, that the geometric solution is applied on two variables only.

Homework (1.2):

HW1:

$$\text{Max } Z = 3x_1 + 2x_2$$

s.t.

$$2x_1 - 3x_2 \leq 6$$

$$x_1 + x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

HW2:

$$\text{Min } Z = 3x_1 - x_2$$

s.t.

$$-3x_1 - 2x_2 \leq 6$$

$$5x_1 + 4x_2 \geq 20$$

$$8x_1 + 3x_2 \leq 24$$

$$x_1, x_2 \geq 0$$