

1.3 Standard Form of a Maximum Problem

Maximum LPP is said to be in standard form (SF) if the following conditions are met:

1. All the variables are nonnegative.
2. All other constraints are written as a linear expression less or equal to a positive constant.

Example (1.11):

(a). $Max Z = 5x_1 + 4x_2$

s.t.

$$3x_1 + 4x_2 \leq 120, x_1 \geq 0$$

$$4x_1 + 3x_2 \leq 20, x_2 \geq 0$$

(b). $Max Z = 8x_1 + 2x_2 + 3x_3$

s.t.

$$4x_1 + 8x_2 \leq 120$$

$$3x_2 + 4x_3 \leq 120,$$

$$x_1, x_2 \geq 0$$

(c). $Max Z = 6x_1 - 8x_2 + x_3$

s.t.

$$3x_1 + x_2 \leq 10$$

$$4x_1 - x_2 \leq 5$$

$$x_1 + x_2 + x_3 \geq -3$$

$$x_1, x_2, x_3 \geq 0$$

(d). $Max Z = 8x_1 + x_2$

s.t.

$$3x_1 + 4x_2 \geq 2$$

$$x_1 + x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Solution:

- (a). The two variables x_1 and x_2 are ≥ 0 , and 2nd condition is hold then the LPP is SF.
- (b). since the variable x_3 is not specified, then the LPP is not in SF.
- (c). The 3rd constraints has negative constant, so its not in SF.
- (d). The 1st constraints has($\geq$) sign, so its not in SF.

Homework (1.3): Examine which maximum LPP's is in SF, if its not mention the reason.

HW1. $Max Z=3x_1 + 4x_2$

s.t. $3x_1 + x_2 \leq 6, x_1 \geq 0$

$$x_1 + 4x_2 \leq 4, x_2 \geq 0$$

HW2. $Max Z=3x_1 + x_2 + x_3$

s.t. $x_1 + x_2 + x_3 \leq 6, x_1 \geq 0$

$$2x_1 + 3x_2 + 4x_3 \leq 0$$

HW3. $Max Z=3x_1 + x_2 + x_3$

s.t. $x_1 + x_2 + x_3 \leq 8, x_1 \geq 0$

$$2x_1 + x_2 + 4x_3 \geq 6, x_2 \geq 0$$

HW4. $Max Z=2x_1 + x_2$

s.t. $x_1 + x_2 \geq -6, x_1 \geq 0$

$$2x_1 + x_2 \leq 4, x_2 \geq 0$$

Notes:

1. In example (11-c), can be modified to be in SF. by multiplying the 3rd constraint by (-1) which be changed to($-x_1 - x_2 - x_3 \leq 3$).
2. Example (11-d), cannot modified to be in SF. even though multiplying the 1st constraint by (-1).

1.4 Slack Variables and the Simplex Tableau

In order to apply the simplex method to maximum problem, we need to first introduce slack variables and construct the initial simplex tableau.

Let us have the following LPP in SF.

$$\text{Max } Z = 3x_1 + 4x_2$$

s.t.

$$2x_1 + 4x_2 \leq 120, x_1 \geq 0$$

$$2x_1 + 2x_2 \leq 80, x_2 \geq 0$$

convert the constraints to equations by adding $S_1, S_2 \geq 0$ them:

$$2x_1 + 4x_2 + S_1 = 120$$

$$2x_1 + 2x_2 + S_2 = 80$$

Where S_1 and S_2 are called slack variables.

Furthermore, the objective function $Z = 3x_1 + 4x_2$ can be rewritten as:

$$-3x_1 - 4x_2 + Z = 0$$

Then the system will be

$$2x_1 + 4x_2 + S_1 = 120$$

$$2x_1 + 2x_2 + S_2 = 80$$

$$-3x_1 - 4x_2 + Z = 0$$

The augmented matrix for this system is:

$$\left(\begin{array}{ccccc|c} x_1 & x_2 & S_1 & S_2 & Z & \\ \hline 2 & 4 & 1 & 0 & 0 & 120 \\ 2 & 2 & 0 & 1 & 0 & 80 \\ -3 & -4 & 0 & 0 & 1 & 0 \end{array} \right)$$

This augmented matrix is called the initial tableau (simplex tableau).

Example (1.12): Introduce the slack variables and setup the initial tableau for this maximum LPP.

$$\text{Max } Z = 3x_1 + 2x_2 + x_3$$

s.t.

$$3x_1 + x_2 + x_3 \leq 30, x_1 \geq 0$$

$$5x_1 + 2x_2 + x_3 \leq 24, x_2 \geq 0$$

$$x_1 + x_2 + 4x_3 \leq 20, \quad x_3 \geq 0$$

Solution:

$$3x_1 + x_2 + x_3 + S_1 = 30$$

$$5x_1 + 2x_2 + x_3 + S_2 = 24$$

$$x_1 + x_2 + 4x_3 + S_3 = 20$$

$$-3x_1 - 2x_2 - x_3 + Z = 0$$

$$\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & S_1 & S_2 & S_3 & Z & B \\ \hline 3 & 1 & 1 & 1 & 0 & 0 & 0 & 30 \\ 5 & 2 & 1 & 0 & 1 & 0 & 0 & 24 \\ 1 & 1 & 4 & 0 & 0 & 1 & 0 & 20 \\ -3 & -2 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}$$

1.4.1 Matrix Notation

Let us have the following Maximum LPP in SF:

$$\text{Max. } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n \quad \dots(4)$$

Subject to:

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \end{array} \right\} \quad \dots(5)$$

$$x_j \geq 0, j=1, \dots, n. \quad \dots(6)$$

If we let:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$C = [c_1 \quad c_2 \quad \dots \quad c_n]$$

We want to find the vector $X \in R^n$ to Maximize the

$$Z = CX \quad \dots(7)$$

s.t.

$$AX \leq B \quad \dots(8)$$

$$X \geq 0 \quad \dots(9)$$

Example (1.13): Example(11-a) can be written as: $X \in R^2$

$$\text{Max. } Z = [5 \quad 4] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 120 \\ 20 \end{bmatrix} \text{ and } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The new LPP with slack variables can be written as:

$$Z = CX \quad \dots(10)$$

s.t.

$$AX = B \quad \dots(11)$$

$$X \geq 0 \quad \dots(12)$$

Where now

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 1 & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 & 0 & \dots & 1 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix}$$

$$\text{and } C = [c_1 \quad c_2 \quad \dots \quad c_n \quad 0 \quad \dots \quad 0]$$

Note: We can express the matrix form in table form of the simplex tableau.