

1.5 Standard Form of a Minimum Problem

Minimum problem is said to be in SF if it satisfies the following conditions:

1. All the variables must be nonnegative.
2. All the other constraints must be written with $\geq$ signs.
3. The objective function to be minimum must be written with non-negative coefficients.

Example (1.17): Determine which of the following minimum LPP is in SF.

(a). $\text{Min } Z = 2x_1 + 3x_2$

s.t.

$$x_1 + 3x_2 \geq 24, \quad x_1 \geq 0$$

$$2x_1 + x_2 \geq 16, \quad x_2 \geq 0$$

(b). $\text{Min } Z = 3x_1 - x_2 + 4x_3$

s.t.

$$3x_1 + x_2 + x_3 \geq 12, \quad x_1 \geq 0$$

$$x_1 + x_2 + x_3 \geq 8, \quad x_2 \geq 0, \quad x_3 \geq 0,$$

(c). $\text{Min } Z = 2x_1 + x_2 + x_3$

s.t.

$$x_1 - 3x_2 + x_3 \leq 12, \quad x_1 \geq 0$$

$$x_1 + x_2 + x_3 \geq 1, \quad x_2 \geq 0, \quad x_3 \geq 0$$

(d). $\text{Min } Z = 2x_1 + x_2 + 3x_3$

s.t.

$$-x_1 - 2x_2 + x_3 \geq -2, \quad x_1 \geq 0$$

$$x_1 + x_2 + x_3 \geq 6, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Solution:

(a). Since all (3) conditions are met This minimum is in SF.

(b). Conditions (1) and (2) are met but (3) is not.

- (c). Conditions (1) and (3) are met but (2) is not, if multiplying constraint (1) by (-1) the problem in SF.
- (d). Three conditions are met, then the problem in SF.

1.5.1 Duality

Consider the following LPP

$$\left. \begin{array}{l} \text{Max. } Z = CX \\ \text{s.t.} \\ AX \leq B \\ X \geq 0 \end{array} \right\} \quad \dots(1)$$

Where A is $m \times n$, B is $m \times 1$, C is $1 \times n$, X is $n \times 1$.

Then:

$$\left. \begin{array}{l} \text{Min. } Z' = B^T Y \\ \text{s.t.} \\ A^T Y \geq C^T \\ Y \geq 0 \end{array} \right\} \quad \dots(2)$$

These problems called dual problems. Problem (1) is called primal Problem and Problem (2) is called dual problem.

Example (1.18): If the problem is:

$$\text{Max. } Z = [3 \quad 4] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 2 & 3 \\ 3 & -1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$$x_1, x_2 \geq 0$$

Then the dual problem is:

$$\text{Min. } Z' = [3 \quad 4 \quad 2] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 2 & 3 & 5 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$y_1, y_2, y_3 \geq 0$$

This problem can be written in a compact matrix, so the primal problem will be:

$$\left(\begin{array}{cc|c} 2 & 3 & 3 \\ 3 & -1 & 4 \\ 5 & 4 & 2 \\ 3 & 4 & 0 \end{array} \right)$$

Then the dual problem will be:

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 3 \\ 3 & -1 & 4 & 4 \\ 3 & 4 & 2 & 0 \end{array} \right)$$

Theorem (1.2):

- (1). The dual of the dual problem is the primal problem.
- (2). The optimal solution (if it exists) of minimum LPP has the same value as the optimal solution of maximum that is its dual.

1.5.2 The Dualization Process

1. Write the minimum problem in SF.
2. Construct the special matrix from the constraints and the objective function.
3. Interchange the rows and columns to form the special matrix of the dual problem.
4. Translate this matrix in to maximum problem in SF.

Example (1.19): Find the dual of min. problem:

$$\text{Min } Z = 2x_1 + 3x_2$$

s.t.

$$2x_1 + x_2 \geq 6, \quad x_1, x_2 \geq 0$$

$$x_1 + 2x_2 \geq 4$$

$$x_1 + x_2 \geq 5$$

Solution: the matrix of the minimum SF problem is:

$$\left(\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 4 \\ 1 & 1 & 5 \\ 2 & 3 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 6 & 4 & 5 & 0 \end{array} \right)$$

Then the dual problem is:

$$\text{Max } Z' = 6y_1 + 4y_2 + 5y_3$$

s.t.

$$2y_1 + y_2 + y_3 \leq 2$$

$$y_1 + 2y_2 + y_3 \leq 3$$

$$y_1, y_2, y_3 \geq 0$$

Find the optimal solution of this problem, the results are shown in table (1.14):

Table (1.14)

↓							
←	BV	y₁	y₂	y₃	S₁	S₂	Sol.
	S₁	2	1	1	1	0	2
	S₂	1	2	1	0	1	3
	Z'	-6	-4	-5	0	0	0
↓							
←	y₁	1	1/2	1/2	1	0	1
	S₂	0	3/2	1/2	0	1	2
	Z'	0	-1	-2	0	0	6
↓							
←	y₃	2	1	1	1	0	2
	S₂	-1	1	0	0	1	1
	Z	4	1	0	5	0	10

$$Z'=10, y_1=0, y_2=0, y_3=2.$$

Then, the minimum value of the objective function in the original problem is: $Z = 10$.

And the values of x_1 and x_2 can be found in the right end of the objective row of the final tableau: $x_1 = 5, x_2 = 0, Z = 10$.

Example (1.20): (Transportation Problem)

The red tomato company cooperates two plants for canning its tomatoes and has two warehouses for storing the finished products until they are purchased by retailers. The company wants to arrange its shipments from the plants to the warehouses so that the requirements of the warehouses is met and shipping costs are kept at a minimum. The schedule shown in following table represents the per case shipping costs from plant to warehouses.

		Warehouse	
		A	B
Plant	I	\$0.25	\$0.18
	II	\$0.25	\$0.14

Each week, plant **I** can produce at most 450 cases and plant **II** can produce no more than 350 cases of tomatoes. Also, each week, warehouses **A** requires at least 300 cases and warehouses **B** requires at least 500 cases. If we represent the number of cases shipped from plant **I** to warehouses **A** by x_1 from plant **I** to warehouses **B** by x_2 and so on, above data can be represented by table (1.5):

table (1.15)

		Warehouse		Max. availability
		A	B	
Plant	I	x_1	x_2	450
	II	x_3	x_4	350
Min. demand		300	500	

Solution: The LPP is stated as follows:

$$\text{Min } Z = 0.25x_1 + 0.18x_2 + 0.25x_3 + 0.14x_4$$

s.t.

$$x_1 + x_2 \leq 450, \quad x_1 \geq 0$$

$$x_3 + x_4 \leq 350, \quad x_2 \geq 0$$

$$x_1 + x_3 \geq 300, \quad x_3 \geq 0$$

$$x_2 + x_4 \geq 500, \quad x_4 \geq 0$$

modify it to SF.

$$-x_1 - x_2 \geq -450$$

$$-x_3 - x_4 \geq -350$$

$$x_1 + x_3 \geq 300$$

$$x_2 + x_4 \geq 500$$

The special matrix is:

$$\left(\begin{array}{cccc|c} -1 & -1 & 0 & 0 & -450 \\ 0 & 0 & -1 & -1 & -350 \\ 0 & 0 & 1 & 0 & 300 \\ 2 & 1 & 0 & 1 & 500 \\ 0.25 & 0.18 & 0.25 & 0.14 & 0 \end{array} \right)$$

The dual matrix is:

$$\left(\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0.25 \\ -1 & 0 & 0 & 1 & 0.18 \\ 0 & -1 & 1 & 0 & 0.25 \\ 2 & -1 & 0 & 1 & 0.14 \\ -450 & -350 & 300 & 500 & 0 \end{array} \right)$$

The dual max. problem is:

$$\text{Max } Z' = -450y_1 - 350y_2 + 300y_3 + 500y_4$$

s.t.

$$-y_1 + y_3 \leq 0.25, \quad y_1 \geq 0.$$

$$-y_1 + y_4 \leq 0.18, \quad y_2 \geq 0.$$

$$-y_2 + y_3 \leq 0.25, \quad y_3 \geq 0.$$

$$-y_2 + y_4 \leq 0.14, \quad y_4 \geq 0.$$

$$y_1, y_2, y_3, y_4 \geq 0$$

Introducing the slack variables, we construct the initial simplex tableau and proceed to solve the dual problem (see table (1.16)):

table (1.16)

↓										
	BV	y₁	y₂	y₃	y₄	S₁	S₂	S₃	S₄	Sol.
	S₁	-1	0	1	0	1	0	0	0	0.25
	S₂	-1	0	0	1	0	1	0	0	0.18
	S₃	0	-1	1	0	0	0	1	0	0.25
←	S₄	0	-1	0	1	0	0	0	1	0.14
	Z	450	350	-300	-500	0	0	0	0	0
↓										
←	S₁	-1	0	1	0	1	0	0	0	0.25
	S₂	-1	1	0	0	0	1	0	-1	0.04
	S₃	0	-1	1	0	0	0	1	0	0.25
	y₄	0	-1	0	1	0	0	0	1	0.14
	Z	450	-150	-300	0	0	0	0	500	70
↓										
	y₃	-1	0	1	0	1	0	0	0	0.25
←	S₂	-1	1	0	0	0	1	0	-1	0.04
	S₃	1	-1	0	0	-1	0	1	0	0
	y₄	0	-1	0	1	0	0	0	1	0.14
	Z	150	-150	0	0	300	0	0	500	145
↓										
	y₃	-1	0	1	0	1	0	0	0	0.25
	y₂	-1	1	0	0	0	1	0	1	0.04
	S₃	0	0	0	0	-1	1	1	-1	0.04
	y₄	0	0	0	1	0	1	0	0	0.18
	Z	0	0	0	0	300	150	0	350	151

The final tableau yields the solution to the original minimum problem.

$$\therefore Z = \$151, x_1 = 300, x_2 = 150, x_3 = 0, x_4 = 350.$$

This means plant **I** should deliver 300 cases to warehouse **A** and 150 cases to warehouse **B**. Plant **II** should deliver 350 cases to warehouse **B** to keep costs at the minimum (\$151).

Note: If the dual of maximum problem has unbounded solution, then the minimum problem has no feasible solution.

Homework (1.6):

(1). Which of the given minimum problem are in SF.

(a) $Min Z = 2x_1 + 3x_2$

s.t.

$$4x_1 - x_2 \geq 2, \quad x_1 \geq 0$$

$$x_1 + x_2 \geq 1, \quad x_2 \geq 0$$

(b). $Min Z = 3x_1 + 5x_2$

s.t.

$$3x_1 - x_2 \geq 4, \quad x_1 \geq 0$$

$$x_1 - 2x_2 \geq 3, \quad x_2 \geq 0$$

(c). $Min Z = 3x_1 + 7x_2 + x_3$

s.t.

$$x_1 + x_3 \leq 6, \quad x_1, x_2, x_3 \geq 0$$

$$2x_1 + x_2 \geq 4$$

(d). $Min Z = x_1 - x_2 + x_3$

s.t.

$$x_1 + x_2 \geq 6, \quad x_1, x_2, x_3 \geq 0$$

$$2x_1 - x_3 \geq 4$$

(2). Write the dual problem for each minimum LPP.

(a) $Min Z = 2x_1 + 3x_2$

s.t.

$$x_1 + x_2 \geq 2, \quad x_1 \geq 0$$

$$2x_1 + 3x_2 \geq 6, \quad x_2 \geq 0$$

(b). $Min \ Z = 3x_1 + x_2 + x_3$

s.t.

$$x_1 + x_2 + x_3 \geq 5, \quad x_1, x_2, x_3 \geq 0$$

$$2x_1 + x_2 \geq 4,$$

(c). $Min \ Z = 3x_1 + 4x_2$

s.t.

$$2x_1 + x_2 \geq 2, \quad x_1, x_2 \geq 0$$

$$2x_1 + x_2 \geq 6, \quad x_2 \geq 0$$

(d). $Min \ Z = 2x_1 + x_2 + x_3$

s.t.

$$2x_1 + x_2 + x_3 \geq 4, \quad x_1, x_2, x_3 \geq 0$$

$$x_1 + 2x_2 + x_3 \geq 6$$

(3). Solve each minimum LPP by using simplex method:

(a). $Min \ Z = 6x_1 + 3x_2$

s.t.

$$x_1 + x_2 \geq 2, \quad x_1, x_2 \geq 0$$

$$2x_1 + 6x_2 \geq 6, \quad x_2 \geq 0$$

(b). $Min \ Z = x_1 + 2x_2 + x_3$

s.t.

$$x_1 - 3x_2 + 4x_3 \geq 12, \quad x_1, x_2, x_3 \geq 0$$

$$3x_1 + x_2 + 2x_3 \geq 10, \quad x_2, x_3 \geq 0$$

$$x_1 - x_2 - x_3 \geq -8, \quad x_3 \geq 0$$