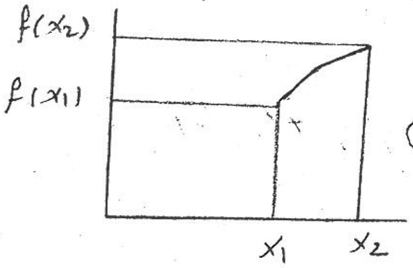
Application of Differentiation

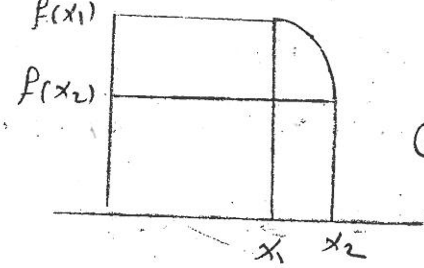
Def:- Let f defined on an interval then:-

1. f is increasing on an interval if for every x_1 and x_2 in interval such that $x_1 < x_2$ we have $f(x_1) < f(x_2)$.

2. f is decreasing on the interval if for every any point in the interval $x_1 < x_2$ we have $f(x_1) > f(x_2)$.



(1)

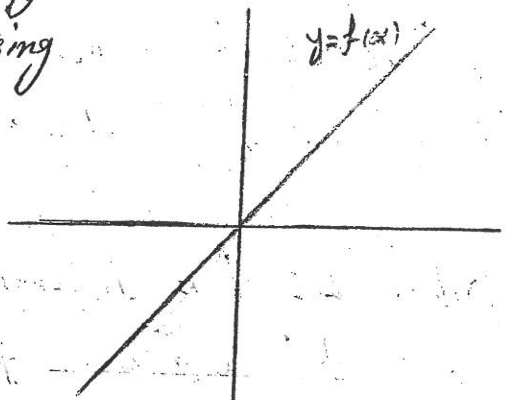


(2)

Theorem:-

a. if $f'(x) > 0$ on (a,b) then f is increasing

b. if $f'(x) < 0$ on (a,b) then f is decreasing

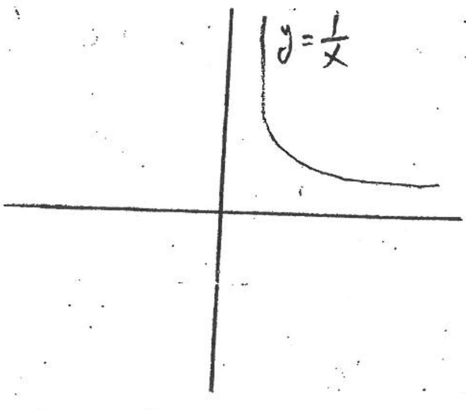


Example (1) $f(x) = x$ is increasing function for all x.

since $x_1 < x_2$ then $f(x_1) < f(x_2)$.

Example (2) let $f : (0, \infty) \to \mathbb{R} \ni f(x) = \frac{1}{x}$ is decreasing fun. for all $x \in (0, \infty)$

since $x_1 < x_2$ then $f(x_1) > f(x_2)$.



Exampl 3 let $f(x) = x^2 - x + 1$

Find point in which the function f is increasing or decreasing

Sol:- $f'(x) = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$- - - - - \frac{1}{2} + + + + +$

$f'(x) > 0$ when $x \in (\frac{1}{2}, \infty)$

$f'(x) < 0$ when $x \in (-\infty, \frac{1}{2})$.

we get: f is increasing in $(\frac{1}{2}, \infty)$

f is decreasing in $(-\infty, \frac{1}{2})$.

Exampl 4 let $f(x) = x^3 - 3x^2 + 1$

Find the interval of increasing and decreasing.

Sol $F'(x) = 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0$

$\Rightarrow x = 0, x = 2$

$- - - - - 0 + + + + +$

$- - - - - - - - - 2 + + +$

$+ + + + 0 - - - - 2 + + +$

incr. dec. incr

$f'(x) > 0$ when $x \in (2, \infty) \cup (0, \infty)$

$f'(x) < 0$ when $x \in (0, 2)$.

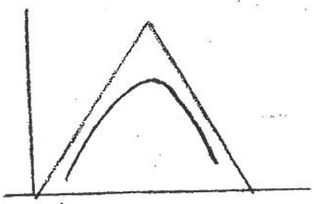
we get f is increasing in $(0, \infty) \cup (2, \infty)$.

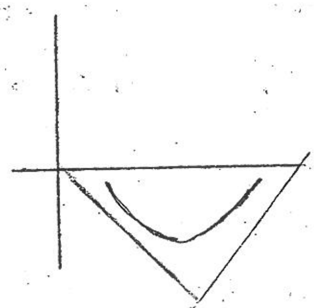
f is dereasing in $(0, 2)$

Def:- Let f is differentiable on open interval (a, b).

1- f is concave up if f' is increasing on (a, b)

2- f is concave down if f' is decreasing on (a, b)





Theorem 1. If $f''(x) > 0$ on (a,b) then concave up $\cup$

2. If $f''(x) < 0$ on (a,b) then concave down $\cap$

Example Find the interval of concavety $f(x) = x^3 - 3x^2 + 1$

Sol $f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6 = 0 \Rightarrow x = 1$.

$f''(x) > 0$ on $(1, \infty)$, $f'' < 0$ on $(-\infty, 1)$

concave down $\cap$ $\cup$

$f''(x)$ – – – – – – (1) + + + + +

concave up

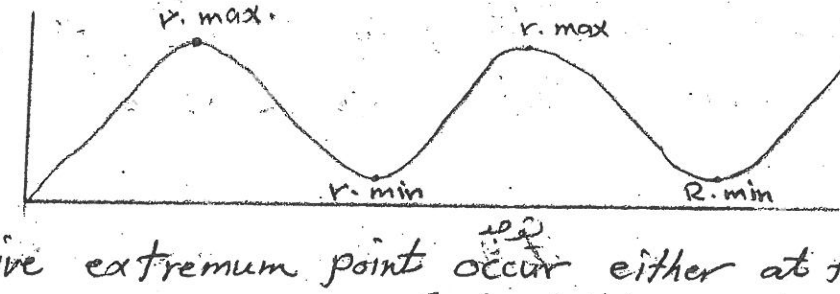
Def:- The point which seperates the interval of concave up and down is called inflection point (نقطة انقلاب)

i-e The point (x_0, y_0) for which $f''(x) = 0$ is called inflection point.

Def:- A function f is said to be relative (نسبية) maximum (عظمى) at x_0 if $f(x_0) \geq f(x)$ For all x in some open interval containing x_0.

Defn:- A function f is said to be relative minimum at x_0 if $f(x_0) \leq f(x)$ For all x in some open interval containing x_0.

Def:- A function f is said to be relative extremum at x_0 if it has either relative max. or relative min. at x_0.



Def:- The relative extremum point occur (توجد) either at the point where the graph of f has horizontal (عمودي) tangent (مماس) or at the point where f is not differentiable.

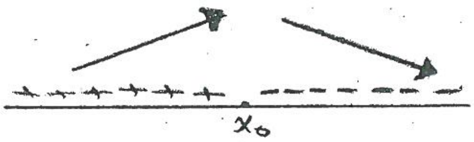
Theorem :- If f has a relative extremum at x_0, then either $f'(x_0) = 0$ or f is not differentiable.

Def :- A critical point for a function f is any value of x in the domain of f at which $f'(x) = 0$ or f is not differentiable.

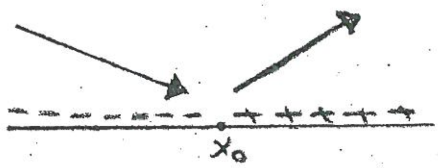
i-e The point (x_0, y_0) for which $f'(x_0) = 0$ is called a critical point

Theorem : suppose f is continuous at a critical point x_0 :-

1. if $f'(x) > 0$ on the left of x_0 and $f'(x) < 0$ on the right of x_0, then f has a relative maximum at x_0. نهاية عظمى



2. if $f'(x) < 0$ on the left of x_0 and $f'(x) > 0$ on the right of x_0 then f has a relative minimum at x_0. نهاية صغرى



Def :- The critical point (x_0, y_0) for which $f''(x_0) < 0$ is called a maximum point.

Def :- The critical point (x_0, y_0) for which $f''(x_0) < 0$ is called a minimum point.

Theorem :- 1. if $f''(x_0) > 0$ Then (x_0, y_0) is called a relative min point.

2. if $f''(x_0) < 0$ then (x_0, y_0) is called a relative max. point

3- if $f''(x_0) = 0$ then First derivative test.

Example :- Locate and describe the relative extrema of :-

1- $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3 = 0 \Rightarrow 3(x^2 - 1) = 0$

$\Rightarrow 3(x-1)(x+1) = 0 \Rightarrow x_0 = \mp 1$

$f''(x) = 6x \Rightarrow f''(1) = 6 > 0 \Rightarrow$ relative min $\Rightarrow (1, -1)$ is relative min point

$f''(-1) = -6 < 0 \Rightarrow$ relative max $\Rightarrow (-1, 3)$ is relative max point

2. $f(x) = x^4 - 2x^2$

$f'(x) = 4x^3 - 4x = 0 \Rightarrow 4x(x^2 - 1) = 0 \Rightarrow 4x = 0$ or $(x^2 - 1) = 0$

$\Rightarrow x = 0$ or $(x-1)(x+1) = 0$

$\Rightarrow x = 0$ or $x = 1$, $x = -1$

$\Rightarrow (0,0), (1,-1), (-1,-1)$ are critical points

$f''(x) = 12x^2 - 4$

$f''(0) = -4 < 0 \Rightarrow$ relative max. $\Rightarrow (0,0)$ is relative max. point

$f''(1) = 12 - 4 = 8 > 0 \Rightarrow$ relative min $\Rightarrow (1, -1)$ is relative min. point

$f''(-1) = 12 - 4 = 8 > 0 \Rightarrow$ relative min $\Rightarrow (-1, -1)$ is relative min point.

Draw the graph of a function $y = f(x) = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$

suppose $y = f(x)$

1- Find $f'(x)$ and $f''(x)$ ١. نجد المشتقة الأولى والثانية

2. Find all x for which $f'(x) > 0$ and for which $f'(x) < 0$

٢. ايجاد مناطق التزايد والتناقص

3. Find all x_0 s.t $f'(x) = 0$ ٣. ايجاد النقاط الحرجة

4. Calculate $f''(x_0) \Rightarrow (x_0, f(x_0))$ is rel. max. or rel. min.

٤. حساب القيم العظمى والصغرى

5. Find x for which $f''(x) = 0 \Rightarrow (x, f(x))$ is inflection point

٥. نساوي المشتقة الثانية بالصفر لنجد نقاط الانقلاب وكذلك مناطق التقعر والتحدب

٦. نحدد بعض النقاط لتحديد الرسم وخاصة نقاط التقاطع مع المحور السيني والصادي

Example sketch the graph $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

sol 1- $y' = \frac{1}{6}(3x^2 - 12x + 9)$

$y'' = \frac{1}{6}(6x - 12)$

2. $f'(x) = 0 \Rightarrow \frac{1}{6}(3x^2 - 12x + 9) = 0$

$\Rightarrow 3x^2 - 12x + 9 = 0$

$\Rightarrow 3(x^2 - 4x + 3) = 0$

$\Rightarrow 3(x-3)(x-1) = 0$

$x = 3, \quad x = 1$

$- - - - - \overset{1}{\cdot} + + + + + +$

$- - - - - - - \overset{3}{\cdot} + + +$

$+ + + + \overset{1}{\cdot} - - - \overset{3}{\cdot} + + +$

inc. dec. inc.

$f'(x) > 0$ when $x \in (3, \infty) \cup (-\infty, 1)$

$f'(x) < 0$ when $x \in (1, 3)$

we get f is increasing in $(3, \infty) \cup (-\infty, 1)$

f is decreasing in $(1, 3)$

3. $(1, f(1)), (3, f(3))$ are critical point

$f(1) = \frac{5}{3} \Rightarrow (1, \frac{5}{3})$
$f(3) = 1 \Rightarrow (3, 1)$ } are critical points

4. $f''(x) = \frac{1}{6}(6x - 12) = x - 2$

$f''(1) = -1 < 0 \Rightarrow$ relative max. $\Rightarrow (1, \frac{5}{3})$ is relative max. point

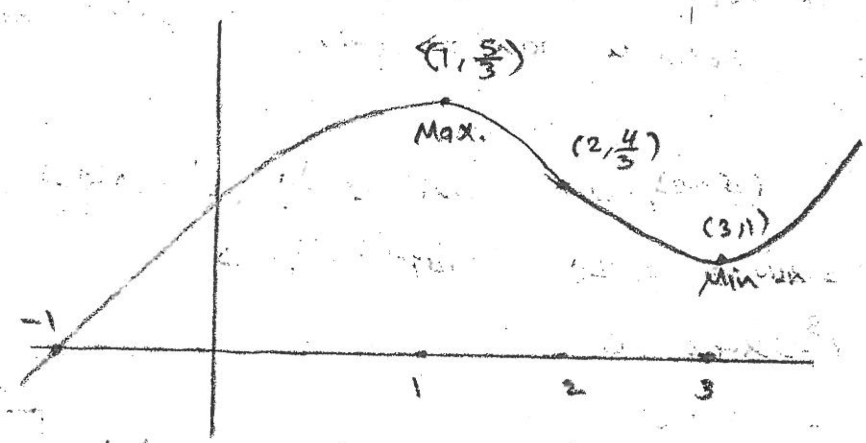
$f''(3) = 1 > 0 \Rightarrow$ relative min $\Rightarrow (3, 1)$ is relative min. point

5. $f''(x) = x - 2 = 0 \Rightarrow x = 2$

$\Rightarrow (2, f(2))$ is inflection point.

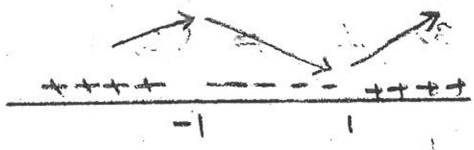
$f(2) = \frac{4}{3}$

we get $(2, \frac{4}{3})$ is inflection point



Ex sketch $y = x^3 - 3x + 2$.

Sol $y' = 3x^2 - 3 = 0 \Rightarrow x^2 - 1 = 0 \Rightarrow (x-1)(x+1) = 0 \Rightarrow x = 1,\ x = -1$



$f'(x) > 0$ when $x \in (1, \infty) \cup (-\infty, -1)$

$f'(x) < 0$ when $x \in (-1, 1)$.

we get f is increasing in $(1, \infty) \cup (-\infty, -1)$

f is decreasing in $(-1, 1)$

$(1, f(1)), (-1, f(-1))$ are critical points

$(1, 0), (-1, 4)$ are critical point.

$f''(x) = 6x$

$f''(1) = 6 > 0 \Rightarrow$ relative min $\Rightarrow (1, 0)$ is relative min point

$f''(-1) = -6 < 0 \Rightarrow$ relative max. $\Rightarrow (-1, 4)$ is relative max. point.

لإيجاد نقاط التقاطع مع المحاور (السيني والصادي).

if $x = 0 \Rightarrow y = 2 \Rightarrow (0, 2)$ نقطة التقاطع مع y-axis

$y = 0 \Rightarrow x^3 - 3x + 2 = 0$

at $x = 1 \Rightarrow 1 - 3 + 2 = 0$

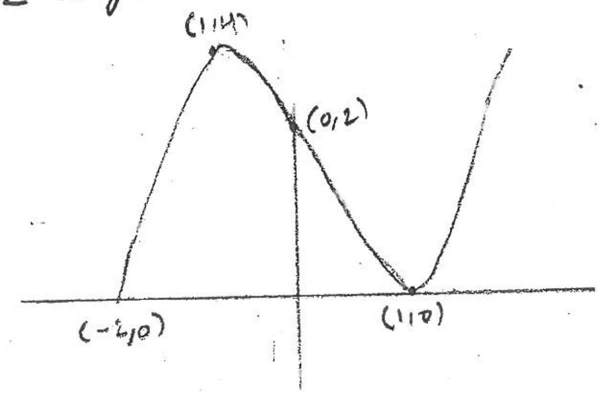
$\therefore x = 1$ is root $\Rightarrow (x - 1)$ factor.

$$
\begin{array}{r|l}
 & x^2 + x - 2 \\
\hline
x - 1 & \cancel{x^3} - 3x + 2 \\
 & \mp x^3 \pm x^2 \\
\hline
 & \cancel{x^2} - 3x \\
 & \mp x^2 \pm x \\
\hline
 & \cancel{-2x + 2} \\
 & \cancel{-2x \mp 2} \\
\hline
 & 0 \quad 0
\end{array}
$$

$(x - 1)(x^2 + x - 2) = 0$

$(x - 1)(x + 2)(x - 1) = 0$

$\left.\begin{array}{l} x = 1 \Rightarrow y = 0 \\ x = -2 \Rightarrow y = 0 \end{array}\right\} \begin{array}{l} (1, 0) \\ (-2, 0) \end{array}$ نقاط التقاطع مع x-axis.



Example Discuss the equation $f(x) = \dfrac{2x}{x^2+1}$

Sol

$$f'(x) = \frac{(x^2+1)\cdot 2 - 2x\cdot 2x}{(x^2+1)^2}$$

$$= \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2}$$

$$f'(x) = 0 \Rightarrow \frac{2-2x^2}{(x^2+1)^2} = 0 \Rightarrow 2-2x^2 = 0 \Rightarrow 2 = 2x^2$$

$$\Rightarrow x^2 = \mp 1 \Rightarrow x_1 = 1,\ x_2 = -1$$

$$f(1) = \frac{2}{1^2+1} = 1, \quad f(-1) = \frac{-2}{(-1)^2+1} = -1$$

$$\Rightarrow f'(x) = \frac{2(1\mp x^2)}{(x^2+1)^2}$$

$1+x$ ` - - - - -1 + + + + + +`

$1-x$ ` + + + + + + 1 - - - -`

` - - - - -1 + + + 1 - - - -`

$f'(x) > 0$ when $x \in (-1,1)$

f is increasing in $(-1,1)$

$f'(x) < 0$ when $x \in (1,\infty) \cup (-\infty,-1)$

f is decreasing in $(1,\infty) \cup (-\infty,-1)$

The points $(1,1), (-1,1)$ are critical point.

$$f''(x) = \frac{(x^2+1)^2(-4x) - (2-2x^2)\cdot 2(x^2+1)\cdot 2x}{(x^2+1)^4} = \frac{4x^3-12x}{(x^2+1)^3}$$

$$f''(1) = \frac{4-12}{(1+1)^3} = \frac{-8}{8} = -1 < 0 \Rightarrow \text{is relative max.}$$

$$f''(-1) = \frac{-4+12}{8} = \frac{8}{8} = 1 > 0 \Rightarrow \text{is relative min.}$$

we get $(1,1)$ is relative max.

$(-1,-1)$ is relative min.

$$f''(x) = \frac{4x^3-12x}{(x^2+1)^3} = 0 \Rightarrow 4x^3-12x=0 \Rightarrow 4x(x^2-3)=0$$

$$4x(x-\sqrt{3})(x+\sqrt{3})=0$$

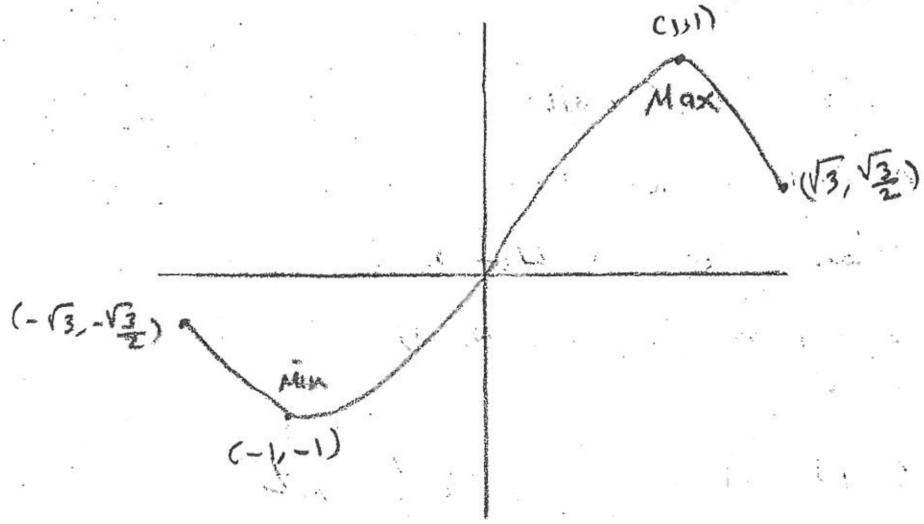
$$x=0,\quad x=\sqrt{3},\quad x=-\sqrt{3}$$

if $x=0 \Rightarrow f(0)=0$

if $x=\sqrt{3} \Rightarrow f(\sqrt{3}) = \frac{2\sqrt{3}}{3+1} = \frac{\sqrt{3}}{2}$

if $x=-\sqrt{3} \Rightarrow f(-\sqrt{3}) = \frac{-2\sqrt{3}}{3+1} = -\frac{\sqrt{3}}{2}$

Then $(0,0)$, $(\sqrt{3}, \frac{\sqrt{3}}{2})$, $(-\sqrt{3}, -\frac{\sqrt{3}}{2})$ are inflection points



محاذي

Def:-- A Line $x = x_0$ is called a vertical a symptote For the graph of a Function f if $f(x) \to +\infty$ or $f(x) \to -\infty$ as $x \to x_0$

$$\boxed{\lim_{x \to x_0} f(x) = \mp\infty \Rightarrow x = x_0 \quad \text{محاذي عمودي}}$$

Def:-- A Line $y = L$ is called horizontal a symptote for the graph of a Function f if $f(x) \to L$ as $x \to \mp\infty$.

$$\boxed{\lim_{x \to \mp\infty} f(x) = L \rightarrow y = L \qquad \text{محاذي افقي}}$$

ملاحظات عن المحاذيات الافقيه

١. اذا كانت اس البسط اكبر من اس المقام فلا يوجد محاذي افقي

٢. اذا كانت اس البسط اقل من اس المقام فالمحاذي الافقي = صفر

٣. اذا كانت اس البسط = اس المقام فالمحاذي الافقي = $\dfrac{\text{معامل اكبر اس في البسط}}{\text{معامل اكبر اس في المقام}}$

Ex let $f(x) = \dfrac{x}{x-2}$

1- $\displaystyle\lim_{x \to 2^+} \frac{x}{x-2} = \frac{2}{0} = \infty \quad , \quad \lim_{x \to 2^-} \frac{x}{x-2} = -\infty$

$\therefore x = 2$ is vertical asymptote.

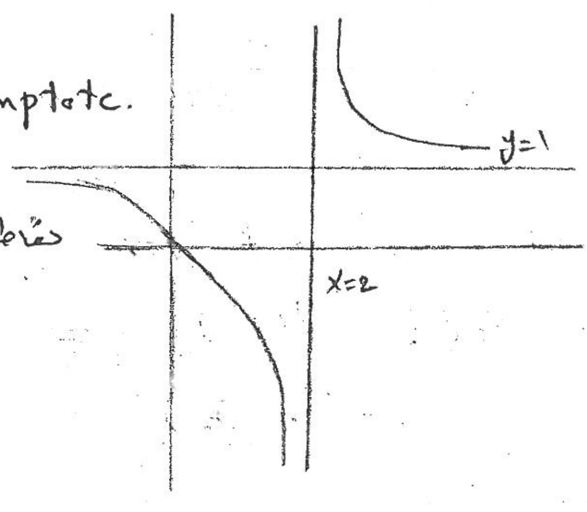
$\displaystyle\lim_{x \to +\infty} \frac{x}{x-2} = \lim_{x \to \infty} \frac{\cancel{x}}{\cancel{x}-2} = 1$

$\therefore y = 1$ is horizontial symptote.

let $x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$

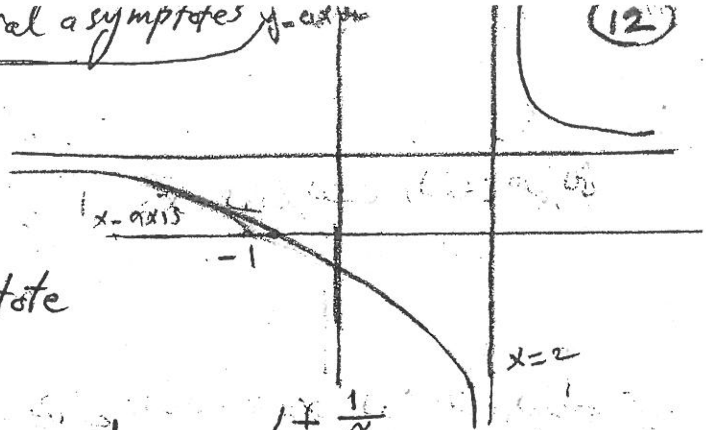
$y = 0 \Rightarrow x = 0 \Rightarrow (0,0)$

نقاط التقاطع مع
x-axis
y-axis



Examples on a vertical and horizentral asymptotes

Ex let $f(x) = \dfrac{x+1}{x-2}$ $\quad y=1$



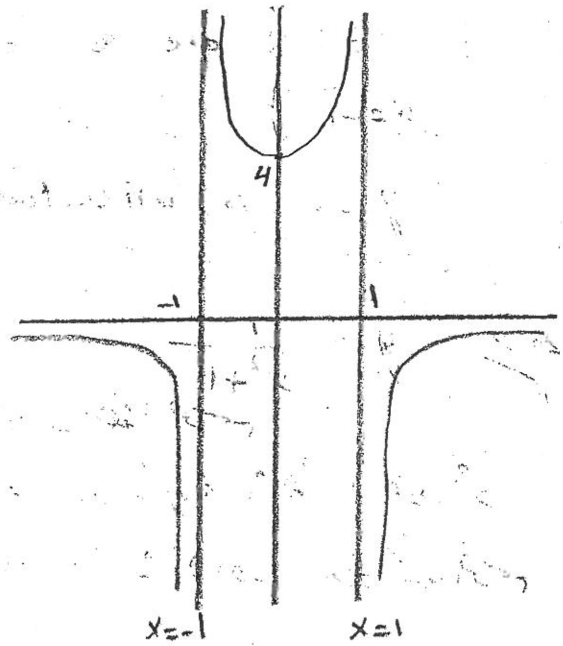
$\therefore$ $x=2$ is vertical asymptote

$$\lim_{x\to\infty}\frac{x+1}{x-2} = \lim_{x\to-\infty}\frac{x+1}{x-2} = \lim_{x\to\mp\infty}\frac{1+\frac{1}{x}}{1-\frac{2}{x}} = 1.$$

$\therefore$ $y=1$ is horizential asymptote.

Ex let $y = \dfrac{4}{1-x^2}$

Sol $y = \dfrac{4}{(1-x)(1+x)}$

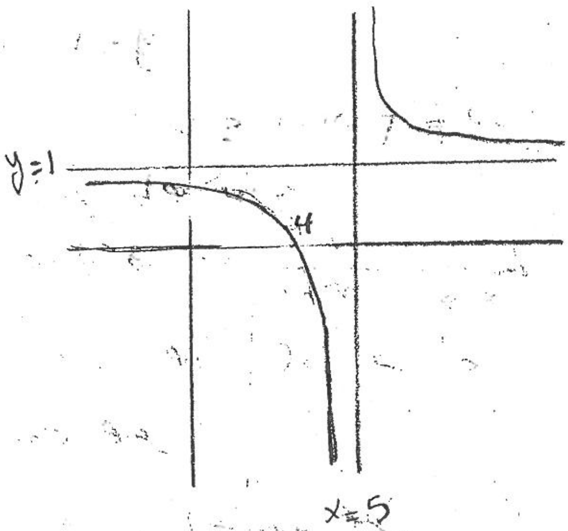


$1-x=0 \Rightarrow x=1$
$1+x=0 \Rightarrow x=-1$ $\Big\}$ are vertical asymptote

$\because$ درجه البسط أقل من درجه المقام $\therefore$ المحاذي الافقي يساوي صفر

$\therefore$ $y=0$

Ex $y = \dfrac{x-4}{x-5}$



Sol $x=5$ is vertical asymptote

$$\lim_{\substack{x\to\infty\\ -\infty}}\frac{x-4}{x-5} = \lim_{x\to\mp\infty}\frac{1-\frac{4}{x}}{1-\frac{5}{x}} = 1$$

$\therefore$ $y=1$ is horizential asymptote

رسم الدوال الكسرية $\frac{P(x)}{Q(x)}$

الشرط الضروري لرسم الدوال الكسرية ان لا يوجد عامل مشترك بين البسط والمقام

١. نفرض $P(x)=0$ لنجد نقاط التقاطع مع محور السينات

٢. نجد المحاذيات العمودية $\lim_{x\to x_0^{\mp}} f(x)=\pm\infty \Rightarrow x=x_0$

٣. نجد المحاذيات الافقية $\lim_{x\to\mp\infty} f(x)=L \Rightarrow y=L$

٤. نجد المشتقة الاولى والثانية لتحديد النقاط الحرجة ونقاط الانقلاب ومناطق التزايد والتناقص والنهايات العظمى والصغرى ومناطق التقعر والتحدب

Example:- sketch the graph of $f(x)=\frac{x^2-1}{x^3}$

Sol :- 1. let $x^2-1=0 \Rightarrow (x-1)(x+1)=0 \Rightarrow x=1,\ x=-1$

$\therefore$ x-intersection is $(1,0)$, $(-1,0)$.

2. $\lim_{x\to 0^+}\frac{x^2-1}{x^3}=-\infty$

$\lim_{x\to 0^-}\frac{x^2-1}{x^3}=\infty$

$\Rightarrow x=0$ المحاذي العمودي

3. $\lim_{x\to\infty}\frac{x^2-1}{x^3}=0 \Rightarrow y=0$ المحاذي الافقي

4. $y'=\frac{x^3\cdot 2x-(x^2-1)\cdot 3x^2}{x^6}=\frac{2x^4-3x^4+3x^2}{x^6}=\frac{2x^2-3x^2+3}{x^4}$

$\frac{-x^2+3}{x^4}=0 \Rightarrow -x^2+3=0 \Rightarrow x^2=3 \Rightarrow x=\mp\sqrt{3}$

if $x=-\sqrt{3} \Rightarrow f(-\sqrt{3})=\frac{-2}{3\sqrt{3}} \Rightarrow (-\sqrt{3},\frac{-2}{3\sqrt{3}})$

$+++\ \ ---\ \ |\ \ ---\ \ +++$

$-\sqrt{3}$ $\quad x=0 \quad$ $\sqrt{3}$

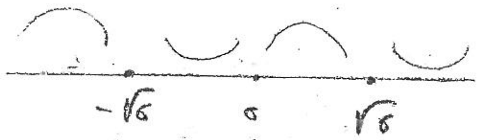
$x=\sqrt{3} \Rightarrow f(\sqrt{3})=\frac{2}{3\sqrt{3}} \Rightarrow (\sqrt{3},\frac{2}{3\sqrt{3}})$

$(-\sqrt{3},\frac{-2}{3\sqrt{3}})$, $(\sqrt{3},\frac{2}{3\sqrt{3}})$ are critical points

$$y'' = \frac{x^4(-2x) - (-x^3+3)\cdot 4x^3}{x^8} = \frac{-2x^5+4x^5-12x^3}{x^8} = \frac{2x^5-12x^3}{x^8}$$

$$= \frac{2x^2-12}{x^5} = 0 \Rightarrow 2x^2-12=0 \Rightarrow 2(x^2-6)=0$$

$$\Rightarrow x^2-6=0 \Rightarrow x = \mp\sqrt{6}$$



$f''(x) = x^2-6$

$$\left.\begin{aligned} &f(\sqrt{3}) = -3 < 0 \Rightarrow \text{relative max} \Rightarrow \left(\sqrt{3}, \tfrac{2}{3\sqrt{3}}\right) \\ &f(-\sqrt{3}) = -3 < 0 \Rightarrow \text{relative max} \Rightarrow \left(-\sqrt{3}, \tfrac{-2}{3\sqrt{3}}\right) \end{aligned}\right\} \text{are relative max-point}$$

$$\left.\begin{aligned} &\text{if } x=\sqrt{6} \Rightarrow f(\sqrt{6}) = \frac{5}{6\sqrt{6}} \\ &x=-\sqrt{6} \Rightarrow f(-\sqrt{6}) = \frac{-5}{6\sqrt{6}} \end{aligned}\right\} \text{are inflection points}$$

Def:- The graph of a function $f(x)$ is said to have a vertical tangent at x_0 if f is continuous at x_0 and $f'(x) \to \mp\infty$ as $x \to x_0$.

$$\lim_{x \to x_0} |f'(x)| = \infty.$$

Example:- Sketch the graph of $f(x) = \sqrt[3]{x}$.

$\frac{dy}{dx}$ + + + + + 0 + + + + +

Sol: $f'(x) = \frac{1}{3} x^{-2/3} = \frac{1}{3x^{2/3}}$ $\Rightarrow$ The function is not defined at $x = 0$

$$f''(x) = \frac{-2}{9} x^{-5/3} = \frac{-2}{9x^{5/3}} = 0$$

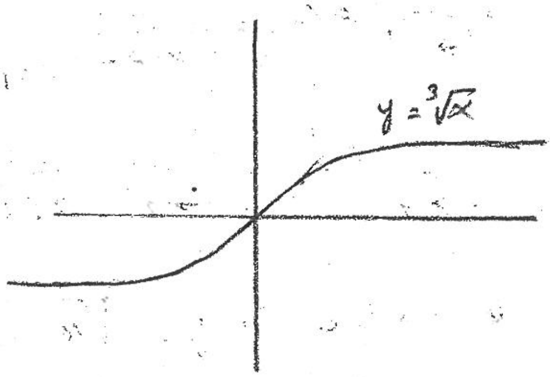
+ + + + + + + 0 − − − − − − − $f''(x)$

let $x = 0 \Rightarrow y = 0 \Rightarrow (0,0)$ is x, y intersection.

نقط التقاطع مع المحورين x, y

$$\lim_{x \to 0} |f'(x)| = \lim_{x \to 0} \left| \frac{1}{3\sqrt[3]{x^2}} \right| = \infty.$$

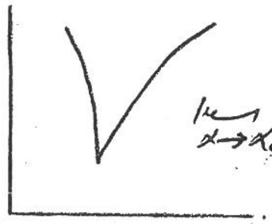
x	-27	-8	-1	0	1	8	27
y	-3	-2	-1	0	1	2	3



Def: The graph of a function f is said to has a cusp $\vee$ $\wedge$ at x_0 if f is continuous at x_0 and $f'(x) \to +\infty$ as $x \to x_0$ from one side and $f'(x) \to -\infty$ as $x \to x_0$ another side.

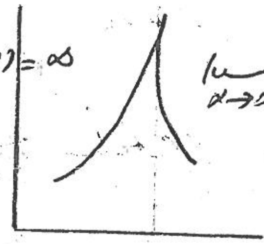
Ex

$\lim\limits_{x \to x_0^-} f'(x) = -\infty$



$\lim\limits_{x \to x_0^+} f'(x) = \infty$

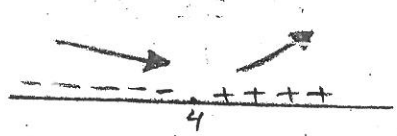
$\lim\limits_{x \to x_0} f(x) = \infty$



$\lim\limits_{x \to x_0^+} f(x) = -\infty$

Ex Sketch the graph of $f(x) = (x-4)^{2/3}$

Sol $f'(x) = \frac{2}{3}(x-4)^{-1/3} = \frac{2}{3(x-4)^{1/3}}$

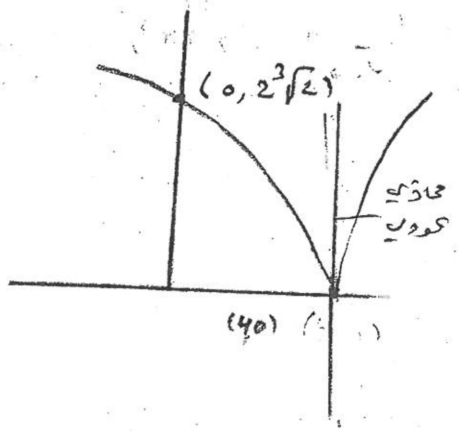


the $f'(x)$ is not defined at $x = 4$

if $x = 4 \Rightarrow y = 0 \Rightarrow (4,0)$ is relative min.

$\lim\limits_{x \to 4^+} f'(x) = \lim\limits_{x \to 4^+} \frac{2}{3(x-4)^{1/3}} = \infty.$

$\lim\limits_{x \to 4^-} f'(x) = \lim\limits_{x \to 4^-} \frac{2}{3(x-4)^{1/3}} = -\infty$



let $x = 0 \Rightarrow y = \sqrt[3]{16} \Rightarrow 2\sqrt[3]{2}$

Extereme - value theorem القيم القصوى

IF a Function f is continuous on $[a,b]$ then f has a maximum value and minimum value of (a,b).

Ex, $f(x) = 2x-1$, $[0,5]$, $[0,5)$, $(0,5]$, $(0,5)$.

sol 1- on $[0,5]$

at $x=0 \Rightarrow (0,-1)$ min

$x=5 \Rightarrow (5,9)$ max.

فقط عندما تكون الفترة مغلقة توجد نهايات صغرى وعظمى

2. on $[0,5)$

at $x=0 \Rightarrow min = -1 \Rightarrow (0,-1)$ min

at $x=5 \Rightarrow \nexists$ max.

3- on $(0,5]$

at $x=0 \Rightarrow \nexists$ min

$x=5 \Rightarrow max = 9$

4- on $(0,5)$

at $x=0 \Rightarrow \nexists$ min

$x=5 \Rightarrow \nexists$ max.

اذا كانت الفترة مفتوحة لا توجد نهايات صغرى وعظمى.

Theorem :- If a function f has exterem value (either max. or min.) on an open interval (a,b) then extreme value occurs of a critical points

كيفيه ايجاد القيم القصوى للداله المستمره f على الفترة المغلقة $[a,b]$

1. نجد النقاط الحرجه للداله f على الفترة (a,b)
2. نحسب قيم الداله f عند النقاط الحرجه وعند نهايات الفترة a and b
3. اكبر القيم في الخطوه 2 هي max. واصغر القيم هي min.

Ex Find the exterem value of $f(x) = 2x^3 - 15x^2 + 36x$ on $[1,5]$

Sol 1. $f'(x) = 6x^2 -$ [illegible] $x^2 + 36$

$$x^2 - 5x + 6 = 0$$

$$(x+3)(x-2) = 0$$

2. $\Rightarrow x = 3 \Rightarrow f(3) = 27$, $x = 2 \Rightarrow f(2) = 28$ } are crtical point

3. $f(1) = 23$, $f(5) = 55$ } حساب قيم الداله عند نهايات الفترة 5,1

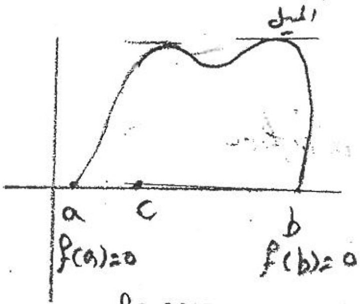
max $= 55$ at $x = 5$

min $= 23$ at $x = 1$

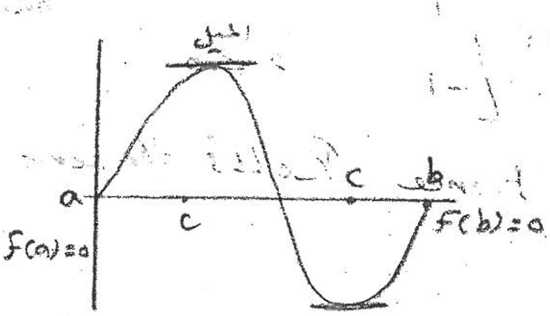
Roll's theorem :-

IF f is differentiable on (a,b) and continuous on $[a,b]$.
IF $f(a) = f(b) = 0$ then there is at Least one number
c between a and b where $f'(c) = 0$ for some c, $a<c<b$

Proof 1- either $f(x) = 0$ $\forall x \in [a,b]$ then $f'(x) = 0$ $\forall x \in (a,b)$
since f is constant on (a,b) thus $f'(c) = 0$ $\forall c \in (a,b)$
or 2- $f(x) \neq 0$ $\forall x \in [a,b]$, then $x \in (a,b)$ $\ni f(x) > 0$ or $f(x) < 0$
if $f(x) > 0$, since f is continuous on $[a,b]$. From Extreme Value theorem, that f has a max. or min. value at some point $c \in [a,b]$. since $f(a) = f(b) = 0$ and $f(x) > 0$
then c can not be an end point. It must lie in (a,b).
f is diff. on (a,b) so, f is diff. at c, so that $f'(c) = 0$

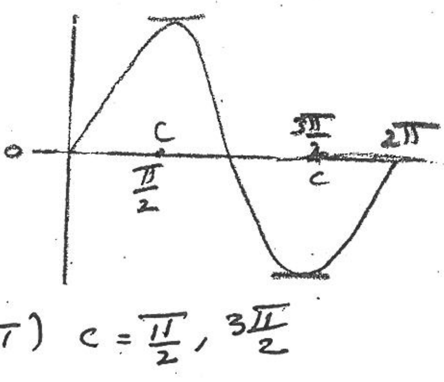


$f(c) > 0$



if $f(x) < 0$ (Exe).

Ex, The Function $f(x) = \sin x$ is cont. and differentiable on $(0, 2\pi)$
$f(0) = 0$, $f(2\pi) = 0$
so f is satisfies Roll's theorem
since $f'(x) = \cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$



$\therefore c = \frac{\pi}{2}, \frac{3\pi}{2}$

i.e There is at least one point c in $(0, 2\pi)$ $c = \frac{\pi}{2}, \frac{3\pi}{2}$
such that $\cos(c) = 0$

Ex $f(x) = |x| - 1$ on $[-1, 1]$

Sol $f(x) = \begin{cases} x-1 & x \geq 0 \\ -x-1 & x < 0 \end{cases}$

1- $f(0) = -1$

2- $\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x-1) = -1$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x-1) = -1$

(3) $\lim_{x \to 0} f(x) = f(0)$

$\therefore$ f is continuous at $x = 0$

$a = -1, \; b = 1$

$f(a) = f(b) = 0$

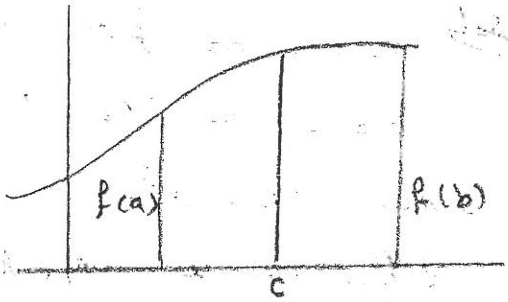
$f'(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ has not derivative at $x = 0$

Hence Roll's theorem is not satisfied.

The Mean Value theorem :-

If $y = f(x)$ is continuous at each point of $[a,b]$ and differentiable on (a,b), then there is at least one number c between a and b ($c \in (a,b)$) For which

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



proof :- Define the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a)$$

$$F(a) = 0 = F(b)$$

Then by Roll's theorem:

$$\exists c \in (a,b) \ni F'(c) = 0$$

$$\Rightarrow F'(x) = f'(x) - \frac{f(b) - f(a)}{b-a}$$

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b-a} = 0$$

we get $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

Ex let $f(x) = x^3+1$
show that f is satisfies Mean value theorem on $[1,2]$
and Find c.

Sol $a = 1 \Rightarrow f(1) = 2$
$b = 2 \Rightarrow f(2) = 9$
since $f(x) = x^3+1$ is polynomail then f is continuous and differentiable everywhere.

Hence $f(x)$ is continuous on $[1,2]$ and differentiable on $(1,2)$

Then f is satisfies the Mean value theorem.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$f'(x) = 3x^2 \Rightarrow f'(c) = 3c^2$$

we get $3c^2 = \dfrac{9-2}{2-1} = 7$

$$c^2 = \frac{7}{3} \Rightarrow c = \mp\sqrt{\frac{7}{3}}$$

Ex Given $f(x) = x^3 - 2x^2 + 3x + 2$ and $a = 0$, $b = 2$
Find all possible value For $x_0 \in (0,2)$ s-t
$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

sol $f(x) = x^3 - 2x^2 + 3x + 2$
$f'(x) = 3x^2 - 4x + 3$
$f'(x_0) = 3(x_0)^2 - 4x_0 + 3$ ---- (1)

$$\frac{F(b) - F(a)}{b - a} = \frac{F(2) - F(0)}{2 - 0} = \frac{4 + 2}{2} = \frac{6}{2} = 3 \text{ --- } (2)$$

(1) = (2)

$3x_0^2 - 4x_0 + 3 = 3$

$3x_0^2 - 4x_0 = 0$

$x_0 (3x_0 - 4) = 0$

we inject $x = 0$ becuse $0 \notin (0,2)$
and we take $3x_0 - 4 = 0 \Rightarrow 3x_0 = 4 \Rightarrow x_0 = \frac{4}{3} \in (0,2)$